

# COMPLEXITY THEORY FOR OPERATORS IN ANALYSIS

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**ABSTRACT.** We propose an extension of the framework for discussing the computational complexity of problems involving uncountably many objects, such as real numbers, sets and functions, that can be represented only through approximation. The key idea is to use (a certain class of) string functions as names representing these objects. These are more expressive than infinite sequences, which served as names in prior work that formulated complexity in more restricted settings. An advantage of using string functions is that we can define their *size* in the way inspired by higher-type complexity theory. This enables us to talk about computation on string functions whose time or space is bounded polynomially in the input size, giving rise to more general analogues of the classes **P**, **NP**, and **PSPACE**. We also define **NP**- and **PSPACE**-completeness under suitable many-one reductions.

Because our framework separates machine computation and semantics, it can be applied to problems on sets of interest in analysis once we specify a suitable representation (encoding). As prototype applications, we consider the complexity of functions (operators) on real numbers, real sets, and real functions. For example, the task of numerical algorithms for solving a certain class of differential equations is naturally viewed as an operator taking real functions to real functions. As there was no complexity theory for operators, previous results only stated how complex the solution can be. We now reformulate them and show that the operator itself is polynomial-space complete.

## 1. INTRODUCTION

*Computable Analysis* [3, 24] studies problems involving real numbers from the viewpoint of computability. Elements of uncountable sets (such as real numbers) are represented by infinite sequences of approximations and processed by Turing machines. This framework is applicable not only to the real numbers but also with great generality to other spaces arising naturally in mathematical analysis. There is a unified way to discuss computability of real functions, sets of real numbers, operators taking real functions as inputs, and so on.

In contrast, the application of this approach to computational complexity has been limited in generality. For example, although there is a widely accepted notion of polynomial-time computable real functions  $f: [0, 1] \rightarrow \mathbb{R}$  on the compact interval that has been studied extensively [16], the same approach does not give a nice class of real functions on  $\mathbb{R}$ . Most of the complexity results in computable analysis to date (with a few exceptions [7, 21, 25]) are essentially limited to the complexity of either real functions with compact domain, or of bounded subsets of  $\mathbb{R}$ . They do not address the complexity of, say, an operator  $F$  that takes real functions  $f: [0, 1] \rightarrow \mathbb{R}$  to another real function  $F(f)$ . There are many positive and negative results [14] about such operators, but typically they are stated in the form

if  $f$  is in the complexity class  $X$ , then  $F(f)$  is in complexity class  $Y$ , and  
there is  $f$  in complexity class  $X$  such that  $F(f)$  is hard for  $Z$ .

More direct statements would be the “uniform” or “constructive” form

the operator  $F$  is in class  $\mathcal{Y}$ , and  
the operator  $F$  is  $\mathcal{Z}$ -hard,

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where  $\mathcal{Y}$  and  $\mathcal{Z}$  are the “higher-order versions” of  $Y$  and  $Z$ . At the level of computability, it is common to ask, as soon as we see a non-uniform result, whether it can be made uniform. For complexity, we cannot even ask this question because we do not know how to formulate  $\mathcal{Y}$  and  $\mathcal{Z}$ . This limitation has been widely recognized; see, for example, [14, pp. 57–58], [25], and [3, p. 484].

To address this problem, we start with the observation (Section 2) that the aforementioned limitation has to do with the fact that traditional formulations of computable analysis do not take into account the “size” of the infinite sequences given to the machine as input. We then propose (Section 3) an extension on the machine model by replacing infinite sequences by what we call *regular functions* on strings. An advantage of using these functions is that we can define their *size* in the way suggested by type-two complexity theory [9, 19]. This enables us to measure the growth of running time (or space) in terms of the input size—exactly what we do in the usual (type-one) complexity theory. We thus obtain the complexity classes analogous to **P**, **NP**, **PSPACE** (and function classes **FP** and **FPSPACE**) by bounding the time or space by *second-order polynomials* in the input size. Analogues of many-one reductions and **NP**- and **PSPACE**-hardness will also be introduced.

We apply this framework to a few specific problems in analysis by using suitable representations of real numbers, real sets, and real functions (Section 4). For real numbers, the induced complexity notions turn out to be equivalent to what has been studied by Ko–Friedman [12] and Hoover [7]. For sets and functions, our approach seems to be the first to provide complexity notions in a unified manner. This is of particular interest, because many numerical problems in the real world are naturally formulated as operators taking sets or functions. For example, consider the operator  $F$  that finds the solution  $F(f)$  of the differential equation (of a certain class) given by a function  $f$ . As mentioned above, the existing non-uniform results [10, 13] only tell us *how complex the solution  $F(f)$  can be when  $f$  is easy*; precisely, they say that if  $f$  is polynomial-time computable,  $F(f)$  is polynomial-space computable and can be polynomial-space hard. But the practical concern for numerical analysis would be *how hard it is to compute  $F$*  (i.e., to compute  $F(f)$  given  $f$ ). We formulate and prove the first result of this kind:  $F$  itself is a polynomial-space complete operator. Our contribution is in introducing the framework making such formulations possible. The technically hard parts of the proofs of the specific results are already done in the proofs of the non-uniform versions, and all we need to do is to check that they uniformize in our sense. The original non-uniform versions are now corollaries of the uniform statements.

**Notation and terminology.** A *multi-valued function* (or *multi-function*)  $F$  from a set  $X$  to a set  $Y$  is formally a subset of  $X \times Y$ . For  $x \in X$ , we write  $F[x]$  for the set of  $y \in Y$  such that  $(x, y)$  belongs to this subset. These  $y$  are the “allowable outputs” on input  $x$ . We denote by  $\text{dom } F$  the set of  $x \in X$  for which  $F[x]$  is nonempty. When  $F[x]$  is a singleton, its unique element is denoted by  $F(x)$ , as usual. If  $F[x]$  is a singleton for all  $x \in \text{dom } F$ , we say that  $F$  is a *partial function*. When in addition  $\text{dom } F = X$ , we say that  $F$  is a *total function*, or simply a *function*.

Like some authors [6, 22], we regard computational tasks (problems) as multi-functions. The classes **FP** and **FPSPACE** consist of multi-functions from strings to strings computed by a machine whose time/space is polynomially bounded. Here, computing a multi-function is to be interpreted according to the “allowable outputs” semantics mentioned above: A machine is said to compute  $F$  if, on any input  $x \in \text{dom } F$ , it outputs *some* element of  $F[x]$ . The classes **FP** and **FPSPACE** that we will define later will also consist of multi-functions.

Note that we do not care what happens on inputs outside  $\text{dom } F$ , unlike some authors who require that such inputs be rejected explicitly. Thus a multi-function can be easy to

compute while having a nasty domain. We also note, however, that allowing  $\text{dom } F$  to be smaller than  $X$  is not so important in the context of time- or space-bounded computation, because a machine that runs past the bound for some inputs can be modified so that it keeps track of the time and outputs an error message when it has run out of time or space.

Throughout the paper,  $\Sigma^*$  denotes the set of finite strings over the alphabet  $\Sigma$ . We will tacitly assume, depending on contexts, that  $\Sigma = \{0, 1\}$  or that  $\Sigma$  contains all symbols appearing in the discussion.

Since our applications mainly involve real numbers, it will be convenient to fix a dense subset of  $\mathbb{R}$  and its encoding. For each  $n \in \mathbb{N}$ , let  $\mathbf{D}_n$  denote the set of strings of the form

$$(1) \quad sx/\underbrace{100 \dots 0}_n,$$

where  $s \in \{+, -\}$  and  $x \in \{0, 1\}^*$ . Let  $\mathbf{D} = \bigcup_{n \in \mathbb{N}} \mathbf{D}_n$ . A string in  $\mathbf{D}$  *encodes* a number in the obvious sense—namely, read (1) as a fraction whose numerator and denominator are integers written in binary with leading zeros allowed. We write  $\llbracket u \rrbracket$  for the number encoded by  $u \in \mathbf{D}$ . The numbers that can be encoded in this way are called *dyadic* numbers.

## 2. TYPE-TWO THEORY OF EFFECTIVITY

There are several equivalent formulations for Computable Analysis. One powerful framework is Weihrauch’s Type-Two Theory of Effectivity (TTE) [3, 24]. In this section, we briefly introduce the infinite sequence model of TTE and discuss some difficulties in dealing with complexity, which motivate our modification in Section 3.

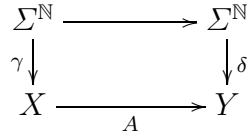
**2.1. Computability.** In the usual computability theory, we use some machine model that computes functions from  $\Sigma^*$  to  $\Sigma^*$ . To discuss computation on other sets  $X$ , we specify an *encoding* of  $X$ —that is, a rule for interpreting an element of  $\Sigma^*$  as an element of  $X$ .

But we want to deal with uncountable sets, such as the set  $\mathbb{R}$  of real numbers. Since the countable set  $\Sigma^*$  cannot encode them, TTE uses the set  $\Sigma^\mathbb{N}$  of *infinite sequences* instead.

Computability of partial functions from  $\Sigma^\mathbb{N}$  to  $\Sigma^\mathbb{N}$  is defined using Turing machines. The machine has an input tape, an output tape, and a work tape, each of which is infinite to the right. We also assume that the output tape is one-way; that is, the only instruction for the output tape is “write  $a \in \Sigma$  in the current cell and move the head to the right”. The difference from the usual setting is in the convention by which the machine reads the input and delivers the output. The input is now an infinite string  $a_0a_1 \dots \in \Sigma^\mathbb{N}$ , and is written on the input tape before the computation starts (with the tape head at the leftmost cell). We say the machine outputs an infinite string  $b_0b_1 \dots \in \Sigma^\mathbb{N}$  if it never halts and writes the string indefinitely on the output tape (that is, for each  $n \in \mathbb{N}$ , it eventually writes  $b_0 \dots b_{n-1}$  into the first  $n$  cells). This defines a class of (possibly partial) computable functions (without any time or space bound) from  $\Sigma^\mathbb{N}$  to  $\Sigma^\mathbb{N}$ . The definition can be extended to multi-functions  $A$ : we say that a machine  $M$  computes  $A$  if  $M$ , on any input  $\varphi \in \text{dom } A$ , always outputs some element of  $A[\varphi]$ .

A *representation*  $\gamma$  of a set  $X$  is formally a partial function from  $\Sigma^\mathbb{N}$  to  $X$  which is surjective—that is, for each  $x \in X$ , there is at least one  $\varphi \in \Sigma^\mathbb{N}$  with  $\gamma(\varphi) = x$ . We say that  $\varphi$  is a  $\gamma$ -*name* of  $x$ . Computability of multi-functions on represented sets is defined as follows.

**Definition 2.1.** Let  $\gamma$  and  $\delta$  be representations of sets  $X$  and  $Y$ , respectively. We say that a machine  $(\gamma, \delta)$ -*computes* a multi-function  $A$  from  $X$  to  $Y$  if it computes the multi-function

FIGURE 1.  $(\gamma, \delta)$ -computing a multi-function  $A$ .

$\delta^{-1} \circ A \circ \gamma$  given by

$$(2) \quad (\delta^{-1} \circ A \circ \gamma)[\varphi] = \begin{cases} \{ \psi \in \text{dom } \delta : \delta(\psi) \in A[\gamma(\varphi)] \} & \text{if } \varphi \in \text{dom } \gamma, \\ \emptyset & \text{otherwise.} \end{cases}$$

In other words, whenever the machine is given a  $\gamma$ -name of an element  $x \in \text{dom } A$ , it must output some  $\delta$ -name of some element of  $A[x]$  (Figure 1).

As an example, we define a representation  $\rho_{\mathbb{R}}$  of the set  $\mathbb{R}$  of real numbers by saying that an infinite string  $\varphi \in \Sigma^{\mathbb{N}}$  is a  $\rho_{\mathbb{R}}$ -name of  $x \in \mathbb{R}$  if  $\varphi$  is of the form  $u_0 \# u_1 \# u_2 \# \dots$  (where  $\#$  is a delimiter symbol not appearing in the  $u_i$ ) such that  $u_i \in \mathbf{D}$  and  $|\llbracket u_i \rrbracket - x| < 2^{-i}$  for each  $i \in \mathbb{N}$ . Thus a real number is specified by a list of rational numbers converging to it. It turns out that  $\rho_{\mathbb{R}}$  is a natural representation with which to discuss computability of real functions. In particular,  $\rho_{\mathbb{R}}$  is *admissible* with respect to the usual topology of  $\mathbb{R}$  [24, Lemma 4.1.6].

To deal with functions of two arguments, we define, for representations  $\gamma$  and  $\delta$  of sets  $X$  and  $Y$ , a representation  $[\gamma, \delta]$  of  $X \times Y$  by  $[\gamma, \delta](a_0 b_0 a_1 b_1 \dots) = (\gamma(a_0 a_1 \dots), \delta(b_0 b_1 \dots))$ .

*Example 2.2.* Addition  $+: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is  $([\rho_{\mathbb{R}}, \rho_{\mathbb{R}}], \rho_{\mathbb{R}})$ -computable. For suppose that we are given names  $\varphi = u_0 \# u_1 \# \dots$  and  $\psi = v_0 \# v_1 \# \dots$  of real numbers  $s$  and  $t$ . An approximation of  $s + t$  with precision  $2^{-m}$ , for each  $m$ , is given by  $\llbracket u_{m+1} \rrbracket + \llbracket v_{m+1} \rrbracket$ .

*Example 2.3.* Multiplication  $\times: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is  $([\rho_{\mathbb{R}}, \rho_{\mathbb{R}}], \rho_{\mathbb{R}})$ -computable. Given names  $\varphi = u_0 \# u_1 \# \dots$  and  $\psi = v_0 \# v_1 \# \dots$  of real numbers  $s$  and  $t$ , let  $k = \max\{|u_0|, |v_0|\}$ . Since  $\llbracket u_0 \rrbracket$  and  $\llbracket v_0 \rrbracket$  are near  $s$  and  $t$ , and it takes more than  $k$  digits to encode a number with absolute value  $\geq 2^k$ , we have  $|s|, |t| < 2^k$ . Hence,  $s \times t$  is approximated with precision  $2^{-m}$  by  $\llbracket u_{m+k+1} \rrbracket \cdot \llbracket v_{m+k+1} \rrbracket$ .

A good thing about the TTE formulation is that, by using suitable representations, we can discuss computation on many other sets. There are often standard ways to obtain representations of higher-type objects such as sets and functions. For example, since we have agreed on the representations  $\rho_{\mathbb{R}}$  of  $\mathbb{R}$ , we can introduce a canonical representation of the set  $C[\mathbb{R}]$  of continuous real functions, and there are reasons to believe that this is the “right” representation [24, Chapter 3].

**2.2. Complexity.** Now we start putting time bounds. This means requiring that the  $n$ th prefix of the output be delivered within time bounded polynomially in  $n$  (and independently of  $\varphi$ ):

**Definition 2.4.** A machine  $M$  runs in *polynomial time* if there is a polynomial  $p$  such that for all  $\varphi \in \Sigma^{\mathbb{N}}$  and  $n \in \mathbb{N}$ , the machine  $M$  on input  $\varphi$  finishes writing the first  $n$  symbols of the output within  $p(n)$  steps. Define *polynomial space* analogously by counting the number of visited cells on all (input, work and output) tapes.

Can we use this notion to define polynomial-time computability of, say, a real function?

2.2.1. *Representations must be chosen carefully.* A little thought shows that the simple combination of Definition 2.4 and the representation  $\rho_{\mathbb{R}}$  is useless [24, Examples 7.2.1, 7.2.3]. On the one hand, the machine  $M$  could “cheat” by writing redundant  $\rho_{\mathbb{R}}$ -names: By writing  $+10000/100000$  instead of  $+1/10$  it gets more time to compute the next approximation. On the other hand, the machine may suffer by receiving redundant names as input, such as the one in which the first approximation is too long to even read in time.

This motivates the use of *signed digit representation*  $\rho_{\text{sd}}$  of  $\mathbb{R}$  [24, Definition 7.2.4] defined as follows, forbidding redundancy carefully:  $\text{dom } \rho_{\text{sd}}$  consists of sequences  $\varphi \in \Sigma^{\mathbb{N}}$  of form  $a_k \dots a_1 a_0 \bullet a_{-1} a_{-2} \dots$  for some  $k$ , where each  $a_i$  is either 0, 1 or  $-1$ , such that  $k = 0$  or  $(a_k, a_{k-1}) \in \{(1, 0), (1, 1), (-1, 0), (-1, -1)\}$ ; if this is the case, set  $\rho_{\text{sd}}(\varphi) = \sum_{i=-\infty}^k a_i \cdot 2^i$ . Thus we read the digit sequence as a binary expansion of a real number (with decimal point  $\bullet$ ) with digits 0, 1 and  $-1$ ; we forbid certain patterns in the first two digits of the integer part in order to exclude redundancy. (See [24, Example 2.1.4.7] for the reason why the usual binary expansion without the “ $-1$ ” symbol does not work.)

Let  $\rho_{\text{sd}}|^{[0,1]}$  denote the restriction of  $\rho_{\text{sd}}$  to (infinite sequences representing) real numbers in  $[0, 1]$ . By Definition 2.4, we know what it means for a real function  $f: [0, 1] \rightarrow \mathbb{R}$  to be *polynomial-time*  $(\rho_{\text{sd}}|^{[0,1]}, \rho_{\text{sd}})$ -computable. This notion turns out to be robust and equivalent to the widely accepted polynomial-time computability of Ko and Friedman [12], so we will drop the prefix “ $(\rho_{\text{sd}}|^{[0,1]}, \rho_{\text{sd}})$ ” from now on. The same goes for *polynomial-space* computability, and for functions on compact intervals or rectangles instead of  $[0, 1]$  (use the pairing function as in Examples 2.2 and 2.3). It is routine to verify that, for example, addition and multiplication  $+, \times: [0, 1] \times [0, 1] \rightarrow \mathbb{R}$  are polynomial-time computable. For more interesting results, see Ko’s book [14], survey [16] or Weihrauch’s book [24, Section 7.3].

2.2.2. *Difficulties in generalizing to other spaces.* Unfortunately, this approach does not extend much further. For example, a naive extension to real functions on  $\mathbb{R}$  (instead of  $[0, 1]$ ) does not work: polynomial-time  $(\rho_{\text{sd}}, \rho_{\text{sd}})$ -computability tends to fail for trivial reasons.

*Example 2.5.* Addition on  $\mathbb{R}$  (Example 2.2) is not polynomial-time  $([\rho_{\text{sd}}, \rho_{\text{sd}}], \rho_{\text{sd}})$ -computable. For suppose that a machine  $([\rho_{\text{sd}}, \rho_{\text{sd}}], \rho_{\text{sd}})$ -computed it within a polynomial time bound  $p$ . In particular, the machine has to write the first symbol of the output in  $t := p(1)$  steps or fewer. Note that this first symbol must be 1 if the sum is greater than 1, and  $-1$  if the sum is less than  $-1$ . In particular, it must be 1 if the two summands are  $2^{t+100}$  and  $-2^{t+50}$ , and  $-1$  if they are  $2^{t+50}$  and  $-2^{t+100}$ . However, the machine cannot tell between these two cases, because it can read at most  $t$  symbols of the input in time.

The trouble seems to be that the time bound is independent of the input. Compare this with the addition of integers (written in binary) by the usual Turing machine. It is in polynomial time, because a large summand would make the “input size” big and thereby give the machine more time. For the same thing to happen for addition of the real numbers, we would need to talk about the “size” of the input and a time bound “polynomial in” it, but we do not have the notion of size for infinite sequences. We could simply require, as Hoover [7] and Takeuti [21] did (see Section 4.1), that the time to output the  $i$ th bit below the decimal point may depend polynomially in both  $i$  and the logarithm of the absolute value of the input real number. This would have the same effect as our proposed extension of the computation model, in this specific case of  $\mathbb{R}$ —but our point is that we want a coherent theory of computation that is applicable to other spaces by just switching representations. There are many objects other than  $\mathbb{R}$  that we want to give representations to. The objects for which the infinite string model gives reasonable notions of complexity are limited, compared to what we can do at the level of computability (see the discussions in Ko [14, pp. 57–58], Weihrauch [25], and Brattka et al. [3, p. 484]). Because of this limitation, the complexity

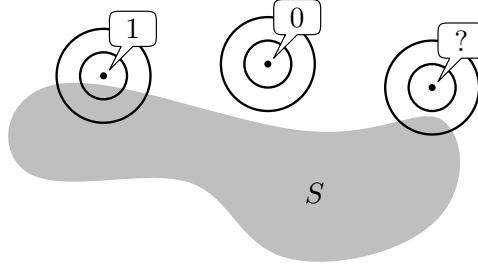


FIGURE 2. Computing a set  $S$  means that, given  $(u, v, 0^n)$ , one can tell whether the distance of the point  $(\llbracket u \rrbracket, \llbracket v \rrbracket)$  from  $S$  is less than  $2^{-n}$  or more than  $2 \cdot 2^{-n}$ .

of operators has been mostly formulated in non-uniform ways. We quote examples of such theorems below. We will reformulate them uniformly later (Theorems 4.6 and 4.10).

**2.2.3. Non-uniform results.** The first pair of results are the positive and negative statements about the operator of taking the convex hull  $CH(S)$  of a closed set  $S \subseteq [0, 1]^2$ .

Polynomial-time computability of a set  $S \subseteq [0, 1]^2$  is defined as follows (see e.g. Braverman [5] for a discussion), using the usual complexity class  $\mathbf{P}$ . We say that  $S$  is *polynomial-time computable* if there is a function  $\varphi: \Sigma^* \rightarrow \{0, 1\}$  in  $\mathbf{P}$  such that, for any  $n \in \mathbb{N}$  and  $u, v \in \mathbf{D}$ ,

- $\varphi(u, v, 0^n) = 1$  if  $\text{dist}(\llbracket u \rrbracket, \llbracket v \rrbracket, S) < 2^{-n}$ , and
- $\varphi(u, v, 0^n) = 0$  if  $\text{dist}(\llbracket u \rrbracket, \llbracket v \rrbracket, S) > 2 \cdot 2^{-n}$ ,

where  $\text{dist}(p, S) := \inf_{q \in S} \|p - q\|$  denotes the Euclidean distance of point  $p \in \mathbb{R}^2$  from  $S$  (Figure 2). Likewise,  $S$  is said to be *nondeterministic polynomial-time computable* if there is such a  $\varphi$  in  $\mathbf{NP}$  (recall the asymmetry between the outcomes 1 and 0 in the definition of  $\mathbf{NP}$ : we require an easily verifiable certificate for  $(\llbracket u \rrbracket, \llbracket v \rrbracket)$  being close to  $S$ ).

Ko and Yu [17] and Zhao and Müller [26] essentially proved<sup>1</sup> the following non-uniform theorems about the complexity of taking the convex hull of a set.

**Theorem 2.6.** *If a closed set  $S \subseteq [0, 1]^2$  is polynomial-time computable, then its convex hull  $CH(S)$  is nondeterministic polynomial-time computable.*

**Theorem 2.7.** *Unless  $\mathbf{P} = \mathbf{NP}$ , there exists a closed set  $S \subseteq [0, 1]^2$  which is polynomial-time computable, but whose convex hull  $CH(S)$  is not.*

For  $A \subseteq \mathbb{R}^d$ , let  $C[A]$  be the set of continuous functions from  $A$  to  $\mathbb{R}$ . The second pair of results concerns the initial value problem (IVP) of the differential equation

$$(3) \quad h(0) = 0, \quad h'(t) = g(t, h(t)),$$

where  $g \in C[[0, 1] \times \mathbb{R}]$  is given and  $h \in C[0, 1]$  is the unknown. It is well known (see [10, beginning of Section 3]) that the solution  $h$  exists and is unique if  $g$  is *Lipschitz continuous* (in the second argument), that is,

$$(4) \quad |g(t, y_0) - g(t, y_1)| \leq L \cdot |y_0 - y_1|$$

<sup>1</sup>Ko and Yu state both the positive and the negative results (Theorems 2.6 and 2.7) for polynomial-time *strong recognizability* instead of our computability [17, Corollaries 4.3 and 4.6, respectively], but their proof almost works for computability as well. For a discussion comparing the two notions, see Braverman [5], where Ko's strong recognizability is called *weak computability*. Zhao and Müller use the polynomial-time computability equivalent to ours and prove Theorem 2.6 [26, Theorem 4.3]. For the positive part, in fact they prove a uniform result essentially equivalent to the positive part of our Theorem 4.6 [26, Theorem 4.1]. Although they state the upper bound of “exponential time”, their proof contains the argument that is necessary to derive the non-uniform  $\mathbf{NP}$  upper bound (our Theorem 2.7) [26, Lemma 4.2].

for some constant  $L$  independent of  $t, y_0, y_1$ . The following results state how complex  $h$  can be, assuming that  $g$  is polynomial-time computable. Since polynomial-time computability is defined only for functions with compact domain, we restrict  $g$  to the rectangle  $[0, 1] \times [-1, 1]$ . If there is a solution  $h \in C[0, 1]$  whose values stay in  $[-1, 1]$  (in which case  $h$  is unique, as mentioned above), we write  $LipIVP(g)$  for this  $h$ . Thus  $LipIVP$  is a partial function from  $CL[[0, 1] \times [-1, 1]]$  to  $C[0, 1]$ , where the former set is the subset of  $C[[0, 1] \times [-1, 1]]$  consisting of Lipschitz continuous functions.

**Theorem 2.8** ([13, Section 4]<sup>2</sup>). *If  $g \in \text{dom } LipIVP$  is polynomial-time computable, then  $LipIVP(g)$  is polynomial-space computable.*

**Theorem 2.9** ([10, Theorem 3.2]). *There is a polynomial-time computable function  $g \in \text{dom } LipIVP$  such that  $LipIVP(g)$  is polynomial-space complete (in the sense defined in [15] or [10]).*

We can derive from Theorem 2.9 a statement of the form similar to Theorem 2.7:

**Corollary 2.10** ([10, Corollary 3.3]). *Unless  $P = PSPACE$ , there is a real function  $g \in \text{dom } LipIVP$  which is polynomial-time computable but  $LipIVP(g)$  is not.*

### 3. USING FUNCTIONS AS NAMES

We present the main definitions for our framework here.

As we have noticed, the limitations of the approach with the infinite sequences in  $\Sigma^{\mathbb{N}}$  have to do with the fact that they do not carry enough information, and in particular their size is not defined. We replace  $\Sigma^{\mathbb{N}}$  with **Reg**, a class of string functions which we will use as names of real numbers, sets and functions<sup>3</sup>.

In Section 3.1, we introduce the class **Reg** of regular functions and define what it means for a machine to compute a multi-function from **Reg** to **Reg**. Section 3.2 defines what it means for such a machine to run in polynomial time or space, thus introducing several complexity classes of multi-functions on **Reg**. In Section 3.3, we define reductions between such multi-functions, and discuss the resulting notions of hardness. Section 3.4 extends this theory of computation on **Reg** to that on other sets  $X$  by using representations of  $X$  by **Reg**.

**3.1. Computation on regular functions.** We say that a (total) function  $\varphi: \Sigma^* \rightarrow \Sigma^*$  is *regular*<sup>4</sup> if it preserves relative lengths of strings in the sense that  $|\varphi(u)| \leq |\varphi(v)|$  whenever  $|u| \leq |v|$ . We write **Reg** for the set of all regular functions. For the rest of this paper, we will be discussing the complexity of multi-functions from **Reg** to **Reg**. The motivation for considering regular functions (rather than all functions from  $\Sigma^*$  to  $\Sigma^*$ ) will be explained in Section 3.2 where we define their lengths.

Now we begin replacing the role of  $\Sigma^{\mathbb{N}}$  (Section 2.1) by **Reg**. This is a generalization, because an infinite string  $a_0a_1 \dots \in \Sigma^{\mathbb{N}}$  can be identified with a regular function  $\varphi \in \mathbf{Reg}$  that

- (a) takes values of length 1, and
- (b) depends only on the length of the argument,

<sup>2</sup>Ko [13] proved Theorems 2.8 and 2.9 with a condition slightly weaker than Lipschitz continuity. On the other hand, Ota et al. [20] show that Theorem 2.9 can be strengthened to yield a continuously differentiable function  $g$ .

<sup>3</sup>Ko's formulation [14] already uses string functions instead of infinite strings, but it does not take full advantage of this extension, because queries to these functions are mostly restricted to unary strings  $0^n$ .

<sup>4</sup>(Note added in May 2013) After the publication of this article in a journal, some authors started using "length-monotone", which is perhaps a more informative and better terminology.

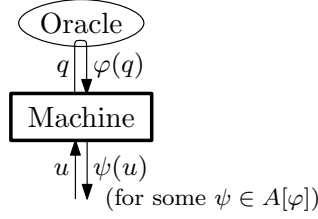


FIGURE 3. A deterministic machine computing a multi-function  $A$  from **Reg** to **Reg**.

by setting  $\varphi(0^n) = a_n$ . In the following, observe that Definitions 3.1.1 and 3.2 extend their counterparts in this sense.

It sometimes makes sense to stop the generalization halfway, removing (b) only and keeping (a). Let **Pred**  $\subseteq$  **Reg** be the set of  $\{0, 1\}$ -valued regular functions.

Instead of the machine that worked on infinite strings, we use an oracle Turing machine (henceforth just “machine”) to convert regular functions to regular functions (Figure 3):

- Definition 3.1.** (1) A deterministic machine  $M$  *computes* a multi-function  $A$  from **Reg** to **Reg** if for any  $\varphi \in \text{dom } A$ , there is  $\psi \in A[\varphi]$  such that  $M$  on oracle  $\varphi$  and any string  $u$  outputs  $\psi(u)$ .
- (2) A nondeterministic machine  $M$  *computes* a multi-function  $A$  from **Reg** to **Pred** if for any  $\varphi \in \text{dom } A$ , there is  $\psi \in A[\varphi]$  such that  $\psi(u) = 1$  if and only if  $M$  on oracle  $\varphi$  and string  $u$  has at least one accepting computation path.

For the precise conventions for issuing and answering queries, follow any of [9, 14, 19].

**3.2. Polynomial time and space.** Recall that regular functions are those that respect lengths in the sense explained at the beginning of Section 3.1. In particular, they map strings of equal length to strings of equal length. Therefore it makes sense to define the *size*  $|\varphi|: \mathbb{N} \rightarrow \mathbb{N}$  of a regular function  $\varphi$  by  $|\varphi|(|u|) = |\varphi(u)|$ . It is a non-decreasing function from  $\mathbb{N}$  to  $\mathbb{N}$ .

Now we want to define what it means for a machine to run in polynomial time. Since  $|\varphi|$  is a function, we begin by defining polynomials “in” a function, following the idea of Kapron and Cook [9]. *Second-order polynomials* (in type-1 variable  $L$  and type-0 variable  $n$ ) are defined inductively as follows: a positive integer is a second-order polynomial; the variable  $n$  is also a second-order polynomial; if  $P$  and  $Q$  are second-order polynomials, then so are  $P + Q$ ,  $P \cdot Q$  and  $L(P)$ . An example is

$$(5) \quad L(L(n \cdot n)) + L(L(n) \cdot L(n)) + L(n) + 4.$$

A second-order polynomial  $P$  specifies a function, which we also denote by  $P$ , that takes a function  $L: \mathbb{N} \rightarrow \mathbb{N}$  to another function  $P(L): \mathbb{N} \rightarrow \mathbb{N}$  in the obvious way. For example, if  $P$  is the above second-order polynomial (5) and  $L(x) = x^2$ , then  $P(L)$  is given by

$$(6) \quad P(L)(x) = ((x \cdot x)^2)^2 + (x^2 \cdot x^2)^2 + x^2 + 4 = 2 \cdot x^8 + x^2 + 4.$$

As in this example,  $P(L)$  is a (usual first-order) polynomial if  $L$  is.

**Definition 3.2.** A (deterministic or nondeterministic) machine  $M$  runs in (*second-order*) *polynomial time* if there is a second-order polynomial  $P$  such that, given any  $\varphi \in \mathbf{Reg}$  as oracle and any  $u \in \Sigma^*$  as input,  $M$  halts within  $P(|\varphi|)(|u|)$  steps (regardless of the nondeterministic choices). Define *polynomial space* analogously by counting the number of visited cells on all (input, work, output and query) tapes.

This extends Definition 2.4, because when  $\varphi$  satisfies (a) (of Section 3.1), the size  $|\varphi|$  is constant, and the bound  $P(|\varphi|)(|u|)$  reduces to a (first-order) polynomial in  $|u|$ .



- Definition 3.3.** (1) We write **FP** (resp. **FPSPACE**) for the class of multi-functions from **Reg** to **Reg** computed by a deterministic machine that runs in second-order polynomial time (resp. space).  
 (2) We write **P** (resp. **NP**) for the class of multi-functions from **Reg** to **Pred** computed by a deterministic (resp. nondeterministic) machine  $M$  that runs in polynomial time.

Note that unlike the type-one counterparts, it is easy to separate **FPSPACE** from **FP** and **NP** from **P**, because the former classes contain functions that depend on exponentially many values of the given oracle.

It is also easy to see that these classes respect the corresponding type-one classes:

- Lemma 3.4.** (1) *Functions in **FP** (resp. **FPSPACE**) map regular functions in **FP** into **FP** (resp. **FPSPACE** into **FPSPACE**).*  
 (2) *Functions in **P** (resp. **NP**) map regular functions in **FP** into **P** (resp. **NP**).*

*Why we use only regular functions.* The idea of using second-order polynomials as a bound on time and space comes from Kapron and Cook’s characterization [9] of Mehlhorn’s class [19] of *polynomial-time computable operators*<sup>5</sup>. This is a class of (total) functionals  $F: (\Sigma^* \rightarrow \Sigma^*) \times \Sigma^* \rightarrow \Sigma^*$ , but they can be regarded as  $F: (\Sigma^* \rightarrow \Sigma^*) \rightarrow (\Sigma^* \rightarrow \Sigma^*)$  by writing  $F(\varphi)(x)$  for  $F(\varphi, x)$ . Kapron and Cook define the size of  $\varphi: \Sigma^* \rightarrow \Sigma^*$  by

$$(7) \quad |\varphi|(n) = \max_{|u| \leq n} |\varphi(u)|, \quad n \in \mathbb{N}.$$

Note that our definition of size for regular  $\varphi$  is a special case of this. They then defined the class of polynomial-time functionals in the same way as Definition 3.3.1. (We added **FPSPACE** by analogy.)

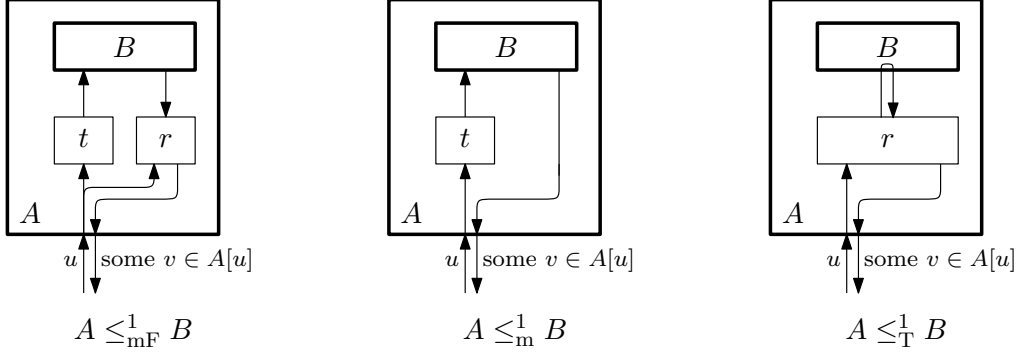
We have restricted attention to regular functions. This is because, in order to obtain reasonable complexity notions, it seems necessary for a machine to be able to tell when the time bound is reached. Note that for usual (type-one) computation, it was easy to find  $|x|$  given  $x$ , and thus to clock the machine with the time bound  $p(|x|)$  for a fixed polynomial  $p$ . In contrast, finding the value (7) for a given  $\varphi$  in general requires exponentially many queries to  $\varphi$ . For regular  $\varphi$ , we can easily find  $|\varphi|(n)$  for each  $n$ , and thus the second-order polynomial  $P(|\varphi|)(|u|)$  is a bound “time-constructible” from  $\varphi$  and  $u$ .

Imposing regularity is hardly a restriction for our purpose, because our intention is to use these functions as names of real numbers, sets and functions, and we can simply require that valid names are those that have been “padded” to be regular. More precisely, there is an efficient machine that takes as oracles a possibly irregular function  $\varphi'$  and a regular function  $\psi$  dominating its length (i.e.,  $|\varphi'(u)| \leq |\psi|(|u|)$  for any string  $u$ ), and delivers a regular function  $\varphi$  such that  $|\varphi| = |\psi|$  and  $\varphi(u) = \varphi'(u) \#\#\dots\#$ . Thus we use  $\varphi$ , instead of  $\varphi'$ , as the name. In many situations we can find such a dominating function  $\psi$ .

For later use we define the pairing function for regular functions as follows (we have been and will be using the tupling functions for strings, which we do not write explicitly): for  $\varphi, \psi \in \mathbf{Reg}$ , define  $\langle \varphi, \psi \rangle \in \mathbf{Reg}$  by setting  $\langle \varphi, \psi \rangle(0u) = \varphi(u)10^{|\psi(u)|}$  and  $\langle \varphi, \psi \rangle(1u) = \psi(u)10^{|\varphi(u)|}$  (we pad the strings to make  $\langle \varphi, \psi \rangle$  regular). Let  $\langle \varphi, \psi, \theta \rangle = \langle \langle \varphi, \psi \rangle, \theta \rangle$ , etc.

**3.3. Reduction and completeness.** Here we define reductions between multi-functions on **Reg** and discuss hardness with respect to these reductions.

<sup>5</sup>Kapron and Cook [9] call them *Basic Feasible Functionals* or *Basic Polynomial-Time Functionals*.

FIGURE 4. Reductions between multi-functions  $A, B$  on  $\Sigma^*$ .

**3.3.1. Reductions.** Recall that the usual many-one reduction between multi-functions  $A, B$  from  $\Sigma^*$  to  $\Sigma^*$  is defined as follows: we say that  $A$  many-one reduces to  $B$  (written  $A \leq_{\text{mF}}^1 B$ ) if there are (total) functions  $r, t \in \mathbf{FP}$  such that for any  $u \in \text{dom } A$ , we have  $r(u, v) \in A[u]$  whenever  $v \in B[t(u)]$ —that is, we have a function  $t$  that converts an input for  $A$  to an input for  $B$ , and another function  $r$  that converts an output of  $B$  to an output of  $A$  (Figure 4, left). The many-one reduction  $\leq_m^1$  between predicates is defined as the special case where we do not convert the output, i.e.,  $r(u, v) = v$  (Figure 4, middle). Since multi-functions over **Reg** also get a function as input, the analogous definition of reductions involves one more converter  $s$ :

**Definition 3.5.** (1) Let  $A$  and  $B$  be multi-functions from **Reg** to **Reg**. We say that  $A$  *many-one reduces* to  $B$  (written  $A \leq_{\text{mF}}^2 B$ ) if there are functions  $r, s, t \in \mathbf{FP}$  such that for any  $\varphi \in \text{dom } A$ , we have  $s(\varphi) \in \text{dom } B$  and, for any  $\theta \in B[s(\varphi)]$ , the function that maps each string  $x$  to  $r(\varphi)(x, \theta(t(\varphi)(x)))$  is in  $A[\varphi]$  (Figure 5, top left).  
 (2) Let  $A$  and  $B$  be multi-functions from **Reg** to **Pred**. We write  $A \leq_m^2 B$  if there are functions  $s, t \in \mathbf{FP}$  such that for any  $\varphi \in \text{dom } A$ , we have  $s(\varphi) \in \text{dom } B$  and, for any  $\theta \in B[s(\varphi)]$ , the function  $\theta \circ t(\varphi)$  is in  $A[\varphi]$  (Figure 5, top right).

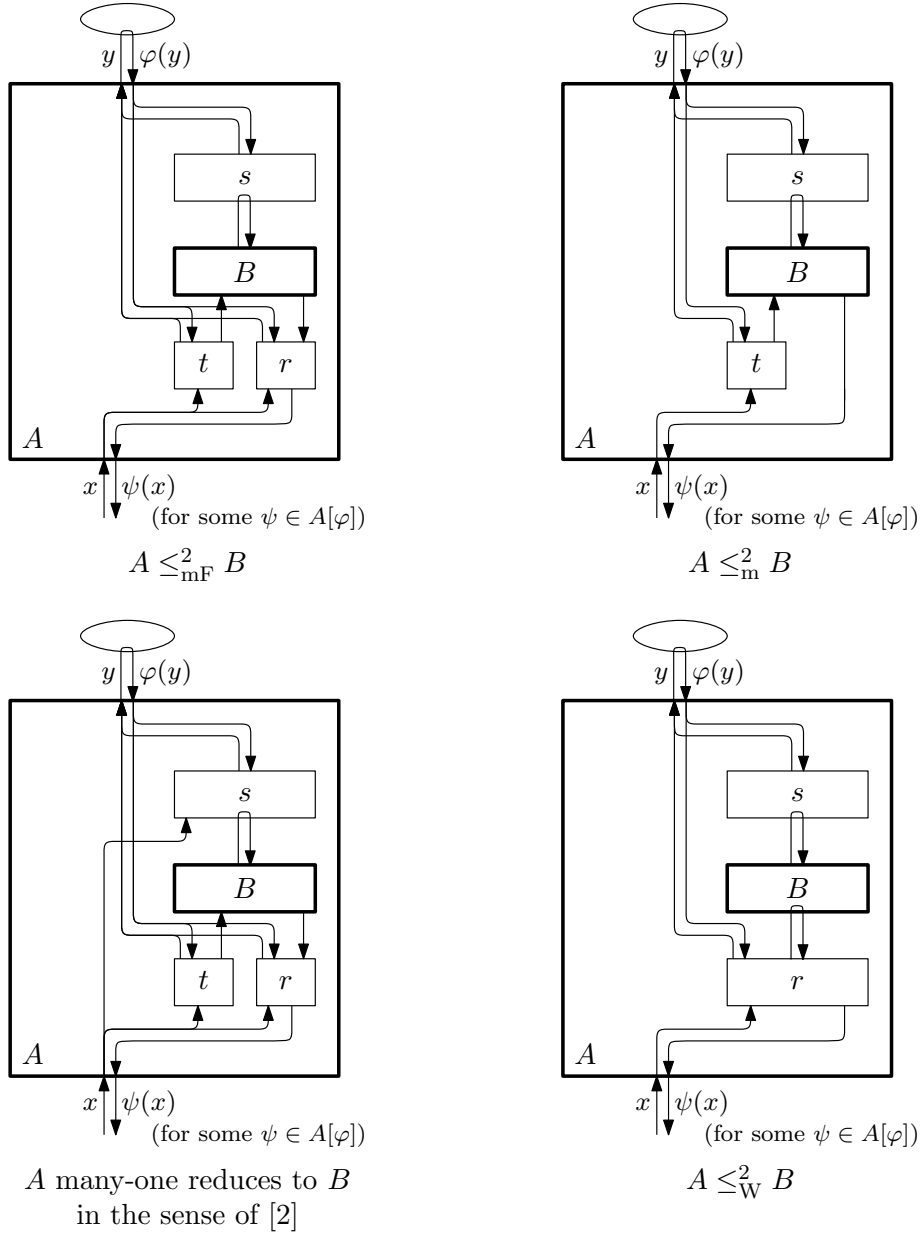
The design of these reductions is somewhat arbitrary. We chose them simply because they are strong enough for our hardness results (Theorems 4.6 and 4.10). What Beame et al. [2] call the “many-one reduction” between type-two functions is slightly stronger than ours in that it passes the string input  $u$  not only to  $t$  and  $r$  but also to  $s$  (Figure 5, bottom left). See the comment after Lemma 3.6 for the reason we did not choose this definition.

Another reasonable notion of reduction is the one on the bottom right of Figure 5:

**Definition 3.5 (continued).** (3) Let  $A$  and  $B$  be multi-functions from **Reg** to **Reg** (or to **Pred**). We say that  $A$  *Weihrauch reduces* to  $B$  (written  $A \leq_W^2 B$ ) if there are functions  $r, s \in \mathbf{FP}$  such that for any  $\varphi \in \text{dom } A$ , we have  $r(\langle \varphi, \psi \rangle) \in A[\varphi]$  whenever  $\psi \in B[s(\varphi)]$ .

This is a polynomial-time version of the continuous reduction used by Weihrauch [23] to compare the degrees of discontinuity of translators between real number representations (see also Brattka and Gherardi [4]). Note that, while this reduction is somewhat analogous to the type-one Turing reduction (Figure 4, right), it also formally resembles the definition of  $\leq_{\text{mF}}^1$ . The many-one reduction  $\leq_{\text{mF}}^2$  is the special case of this reduction  $\leq_W^2$  where  $r$  can query  $\psi$  only once. Beame et al. [2] define an even stronger “Turing reduction”.

In this paper, we will formulate our hardness results mainly using  $\leq_{\text{mF}}^2$  and  $\leq_m^2$  because they give stronger results than  $\leq_W^2$  would. The advantage and disadvantage of this choice will be discussed in Section 4.3 after Theorem 4.10 and Corollary 4.11.

FIGURE 5. Reductions between multi-functions on  $\mathbf{Reg}$ .

**3.3.2. Hardness.** Now that we have the classes (Definition 3.3) and reductions (Definition 3.5), we can talk about hardness. A multi-function  $B$  from  $\mathbf{Reg}$  to  $\mathbf{Reg}$  is **FPSPACE**- $\leq_{\text{mF}}^2$ -hard if  $A \leq_{\text{mF}}^2 B$  for every  $A \in \mathbf{FPSPACE}$ . It is said to be **FPSPACE**- $\leq_{\text{mF}}^2$ -complete if moreover it is in **FPSPACE**. We define **NP**- $\leq_{\text{m}}^2$ -hardness of multi-functions from  $\mathbf{Reg}$  to  $\mathbf{Pred}$  similarly (note that **NP** is not closed under  $\leq_{\text{mF}}^2$ ).

The following lemma states roughly that a  $\mathbf{C}$ - $\leq^2$ -hard multi-function  $B$  maps some function  $\psi \in \mathbf{FP} \cap \mathbf{Reg}$  to a  $\mathbf{C}$ - $\leq^1$ -hard function, where  $\mathbf{C}$  and  $\leq^1$  are the type-one versions of the class  $\mathbf{C}$  and the reduction  $\leq^2$ . But since  $B[\psi]$  may consist of more than one function, we need to assert hardness for the multi-function  $\bigcup(B[\psi])$  defined as follows: for a nonempty set  $F$  of (single-valued total) functions from  $X$  to  $Y$ , we write  $\bigcup F$  to mean the multi-function from  $X$  to  $Y$  defined by  $(\bigcup F)[x] = \{f(x) : f \in F\}$ . Saying that the multi-function  $\bigcup F$  is hard is a stronger claim than saying that each of the functions in  $F$  is hard, because the former

requires that one reduction work for all functions in  $F$ . We need to state the following lemma in this stronger form in order to derive Lemma 3.12 later.

**Lemma 3.6.** (1) *Let  $B$  be an  $\mathbf{FPSPACE}\text{-}\leq_{\mathbf{mF}}^2$ -complete multi-function from  $\mathbf{Reg}$  to  $\mathbf{Reg}$ . Then there is  $\psi \in \text{dom } B \cap \mathbf{FP}$  such that  $\bigcup(B[\psi])$  is  $\mathbf{FPSPACE}\text{-}\leq_{\mathbf{mF}}^1$ -complete.*  
 (2) *Let  $B$  be an  $\mathbf{NP}\text{-}\leq_{\mathbf{m}}^2$ -complete multi-function from  $\mathbf{Reg}$  to  $\mathbf{Pred}$ . Then there is  $\psi \in \text{dom } B \cap \mathbf{FP}$  such that  $\bigcup(B[\psi])$  is  $\mathbf{NP}\text{-}\leq_{\mathbf{m}}^1$ -complete.*

*Proof.* We only prove the first claim (the second claim is similar). There is a function  $A \in \mathbf{FPSPACE}$  that maps some function  $\varphi \in \mathbf{FP} \cap \mathbf{Reg}$  to an  $\mathbf{FPSPACE}\text{-}\leq_{\mathbf{mF}}^1$ -complete function. Since  $A \leq_{\mathbf{mF}}^2 B$ , there are functions  $r, s, t \in \mathbf{FP}$  as in Definition 3.5. By Lemma 3.4, we have  $r(\varphi), s(\varphi), t(\varphi) \in \mathbf{FP}$ . Let  $\psi = s(\varphi)$ . Since  $r(\varphi)$  and  $t(\varphi)$  give a reduction  $A(\varphi) \leq_{\mathbf{mF}}^1 \bigcup(B[\psi])$ , and  $A(\varphi)$  is  $\mathbf{FPSPACE}\text{-}\leq_{\mathbf{mF}}^1$ -complete,  $\bigcup(B[\psi])$  is also  $\mathbf{FPSPACE}\text{-}\leq_{\mathbf{mF}}^1$ -complete.  $\square$

We note that Lemma 3.6 would not have been true, if in the definition of reductions we had fed  $s$  with the string input as Beame et al. [2] do (see the comment after Definition 3.5). For let  $\leq_*^2$  be the reduction which is like  $\leq^2$  but feeds  $s$  with the string input, and let  $B$  be a  $\mathbf{C}\text{-}\leq_*^2$ -complete multi-function. Then the multi-function  $B'$  defined by

$$(8) \quad \text{dom } B' = \{ \langle \text{const}_u, \varphi \rangle : u \in \Sigma^*, \varphi \in \text{dom } B \},$$

$$(9) \quad B'[\langle \text{const}_u, \varphi \rangle] = \{ \text{const}_{\psi(u)} : \psi \in B[\varphi] \},$$

where  $\text{const}_u \in \mathbf{Reg}$  denotes the constant function with value  $u$ , is  $\mathbf{C}\text{-}\leq_*^2$ -complete by the modified reduction where “ $s$  does the job that  $t$  used to do”. Yet each one of the values of  $B'$  is a constant function.

**3.3.3. Some complete problems.** Let  $\mathbf{PSPACE}$  be the subclass of  $\mathbf{FPSPACE}$  consisting of multi-functions from  $\mathbf{Reg}$  to  $\mathbf{Pred}$ . Here we list some  $\mathbf{NP}$ - and  $\mathbf{PSPACE}\text{-}\leq_{\mathbf{m}}^2$ -complete problems. Their completeness can be proved by relativizing the well-known  $\mathbf{NP}$ - and  $\mathbf{PSPACE}\text{-}\leq_{\mathbf{m}}^1$ -completeness in a straightforward way.

We begin with  $\mathbf{NP}\text{-}\leq_{\mathbf{m}}^2$ -complete problems. For a non-decreasing function  $\mu: \mathbb{N} \rightarrow \mathbb{N}$ , define  $\bar{\mu} \in \mathbf{Reg}$  by  $\bar{\mu}(u) = 0^{\mu(|u|)}$ . A Boolean formula involving predicate symbol  $\mathbf{p}$  is an expression built up inductively from Boolean variables  $a_1, a_2, \dots$  using the connectives  $f_1 \wedge f_2, f_1 \vee f_2, \neg f_1$  and  $\mathbf{p}(f_1, \dots, f_n)$  (the arity  $n$  can vary) for any previously obtained formulas  $f_1, f_2, \dots$ .

**Lemma 3.7.** *The following partial functions  $\text{NTIME}^2$ ,  $\text{EXIST}^2$  and  $\text{SAT}^2$  from  $\mathbf{Reg}$  to  $\mathbf{Pred}$  are  $\mathbf{NP}\text{-}\leq_{\mathbf{m}}^2$ -complete:*

- $\text{dom } \text{NTIME}^2$  consists of all triples  $\langle M, \bar{\mu}, \varphi \rangle$  such that  $M$  is a (program of a) non-deterministic (oracle Turing) machine,  $\mu: \mathbb{N} \rightarrow \mathbb{N}$  is non-decreasing,  $\varphi \in \mathbf{Reg}$ , and for any string  $u$ , all computation paths of  $M^\varphi(u)$  halt in time  $\mu(|u|)$  (this  $M$  is a string, so we encode it as the constant function taking this string as value). For any such triple and a string  $u$ , we have  $\text{NTIME}^2(\langle M, \bar{\mu}, \varphi \rangle)(u) = 1$  if and only if  $M^\varphi(u)$  has an accepting path.
- $\text{dom } \text{EXIST}^2 = \mathbf{Pred}$ . For any  $p \in \mathbf{Pred}$ ,  $u \in \Sigma^*$  and  $n \in \mathbb{N}$ , we have  $\text{EXIST}^2(p)(u, 0^n) = 1$  if and only if there is a string  $v$  of length  $n$  with  $p(u, v) = 1$ .
- $\text{dom } \text{SAT}^2 = \mathbf{Pred}$ . For any  $p \in \mathbf{Pred}$  and any string  $u$ , we have  $\text{SAT}^2(p)(u) = 1$  if and only if  $u$  is a Boolean formula involving a predicate symbol  $\mathbf{p}$  and it is made satisfiable when  $\mathbf{p}$  is interpreted as  $p$ .

If  $\varphi$  is a Boolean formula involving predicate symbol  $\mathbf{p}$ , then an expression of the form

$$(10) \quad Q_1 a_1 \cdot Q_2 a_2 \cdot Q_3 a_3 \dots Q_k a_k \cdot \varphi(a_1, \dots, a_k),$$

where each  $Q_i$  is either  $\forall$  or  $\exists$ , is called a *quantified Boolean formula involving predicate symbol  $p$* . Its truth value is determined in the obvious way relative to the predicate to be substituted into  $p$ .

**Lemma 3.8.** *The following partial functions  $\text{SPACE}^2$ ,  $\text{POWER}^2$ ,  $\text{QBF}^2$  from  $\mathbf{Reg}$  to  $\mathbf{Pred}$  are  $\mathbf{PSPACE}\text{-}\leq_m^2$ -complete:*

- $\text{dom } \text{SPACE}^2$  consists of all triples  $\langle M, \bar{\mu}, \varphi \rangle$  such that  $M$  is a (program of a) deterministic (oracle Turing) machine,  $\mu: \mathbb{N} \rightarrow \mathbb{N}$  is non-decreasing,  $\varphi \in \mathbf{Reg}$ , and for any string  $u$ , the computation  $M^\varphi(u)$  uses no more than  $\mu(|u|)$  tape cells and either accepts or rejects (this  $M$  is a string, so we encode it as the constant function taking this string as value). For any such triple and a string  $u$ , we have  $\text{SPACE}^2(\langle M, \bar{\mu}, \varphi \rangle)(u) = 1$  if and only if  $M^\varphi(u)$  accepts.
- $\text{dom } \text{POWER}^2$  consists of all  $f \in \mathbf{Reg}$  that are length-preserving (i.e.,  $|f| = \text{id}$ ). For any such  $f$  and a string  $u$ , we have  $\text{POWER}^2(f)(u) = 1$  if and only if  $f^{2^{|u|}}(u) = 0^{|u|}$ , where we write  $f^k$  for  $f$  iterated  $k$  times.
- $\text{dom } \text{QBF}^2 = \mathbf{Pred}$ . For any  $p \in \mathbf{Pred}$  and any string  $u$ , we have  $\text{QBF}^2(p)(u) = 1$  if and only if  $u$  is a quantified Boolean formula involving a predicate symbol and it is made true by  $p$ .

**3.4. Representations.** As we replaced  $\Sigma^\mathbb{N}$  by  $\mathbf{Reg}$ , we extend the notion of representations accordingly: a *representation*  $\gamma$  of a set  $X$  is a surjective partial function from  $\mathbf{Reg}$  to  $X$ . Computation relative to representations is again formulated by Definition 2.1. This defines what it means for a multi-function  $F$  from  $X$  to  $Y$ , where  $X$  and  $Y$  are sets equipped with representations  $\gamma$  and  $\delta$ , respectively, to be in  $(\gamma, \delta)\text{-}\mathbf{C}$ , where  $\mathbf{C}$  is one of the classes we have defined, such as  $\mathbf{FP}$  and  $\mathbf{FSPACE}$ . This  $\mathbf{C}$  can be  $\mathbf{P}$  or  $\mathbf{NP}$  if  $\text{dom } \delta \subseteq \mathbf{Pred}$ . Also, we say that  $F$  is  $(\gamma, \delta)\text{-}\mathbf{C}\text{-}\leq\text{-hard/complete}$  (for  $\mathbf{C} = \mathbf{FSPACE}, \mathbf{NP}$ ) if  $\delta^{-1} \circ F \circ \gamma$  (see Definition 2.1) is  $\mathbf{C}\text{-}\leq\text{-hard/complete}$ .

**3.4.1. Translation and equivalence.** Here we discuss how the class  $(\gamma, \delta)\text{-}\mathbf{C}$  and  $(\gamma, \delta)\text{-}\mathbf{C}\text{-}\leq\text{-hardness}$  depend on the choice of representations  $\gamma$  and  $\delta$ . For two representations  $\delta$  and  $\delta'$  of the same set, we write  $\delta \leq \delta'$  if there is a function  $F \in \mathbf{FP}$  that *translates*  $\delta$  to  $\delta'$  in the sense that for all  $\varphi \in \text{dom } \delta$ , we have  $F(\varphi) \in \text{dom } \delta'$  and  $\delta(\varphi) = \delta'(F(\varphi))$ . Thus  $\delta$  is “more informative” or “less generic” than  $\delta'$ . It is easy to see the following.

**Lemma 3.9.** *Let  $\mathbf{C}$  be either  $\mathbf{FP}$  or  $\mathbf{FSPACE}$ . Let  $\gamma$  and  $\gamma'$  be representations of a set  $A$ , and  $\delta$  and  $\delta'$  be representations of a set  $B$ . If  $\gamma' \leq \gamma$  and  $\delta \leq \delta'$ , then  $(\gamma, \delta)\text{-}\mathbf{C} \subseteq (\gamma', \delta')\text{-}\mathbf{C}$ .*

We write  $\gamma \equiv \gamma'$  if  $\gamma \leq \gamma'$  and  $\gamma' \leq \gamma$ . Lemma 3.9 implies that the class  $(\gamma, \delta)\text{-}\mathbf{FP}$  is invariant under replacing  $\gamma$  or  $\delta$  with  $\equiv$ -equivalent representations.

By reversing the directions of the translations between  $\gamma, \gamma'$  and  $\delta, \delta'$  in the assumption, we get the implication between hardness results under different representations:

**Lemma 3.10.** *Let  $\gamma$  and  $\gamma'$  be representations of a set  $A$ , and  $\delta$  and  $\delta'$  be representations of a set  $B$ . If  $\gamma \leq \gamma'$  and  $\delta' \leq \delta$ , then a  $(\gamma, \delta)\text{-}\mathbf{FSPACE}\text{-}\leq_W^2\text{-hard}$  multi-function is  $(\gamma', \delta')\text{-}\mathbf{FSPACE}\text{-}\leq_W^2\text{-hard}$ .*

Here,  $\leq_W^2$  is the stronger reduction in Definition 3.5.3 (note that  $\leq_{mF}^2$  would not work).

**3.4.2. Uniform and non-uniform statements.** We say that an element  $x \in X$  is in  $\gamma\text{-}\mathbf{C}$  (where  $\mathbf{C}$  is a usual complexity class of string functions, such as  $\mathbf{FP}$  and  $\mathbf{FSPACE}$ ) if it has a  $\gamma$ -name in  $\mathbf{C}$ . It is  $\gamma\text{-}\mathbf{C}\text{-}\leq\text{-complete}$  (where  $\leq$  is either  $\leq_{mF}^1, \leq_m^1$  or  $\leq_T^1$ ) if  $\bigcup(\gamma^{-1}[x])$  (where  $\cdot^{-1}$  is the inverse image, and  $\bigcup$  is defined before Lemma 3.6) is  $\mathbf{C}\text{-}\leq\text{-complete}$ . Lemmas 3.4 and 3.6 yield the following.

**Lemma 3.11.** *Let  $\gamma$  and  $\delta$  be representations of sets  $X$  and  $Y$ , respectively.*

- (1) *A partial function  $F \in (\gamma, \delta)$ -**FP** maps elements of  $\gamma$ -**FP**  $\cap \text{dom } F$  into  $\delta$ -**FP**. Similarly for **FPSPACE** and **FPSPACE** replacing **FP** and **FP**.*
- (2) *Suppose that  $\text{dom } \delta \subseteq \mathbf{Pred}$ . Then a partial function  $F \in (\gamma, \delta)$ -**P** maps elements of  $\gamma$ -**FP**  $\cap \text{dom } F$  into  $\delta$ -**P**. Similarly for **NP** and **NP** replacing **P** and **P**.*

**Lemma 3.12.** *Let  $\gamma$  and  $\delta$  be representations of sets  $X$  and  $Y$ , respectively.*

- (1) *A  $(\gamma, \delta)$ -**FPSPACE**- $\leq_{\text{mF}}^2$ -complete partial function  $F$  maps some element of  $\gamma$ -**FP**  $\cap \text{dom } F$  to a  $\delta$ -**FPSPACE**- $\leq_{\text{mF}}^1$ -complete element of  $Y$ .*
- (2) *Suppose that  $\text{dom } \delta \subseteq \mathbf{Pred}$ . Then a  $(\gamma, \delta)$ -**NP**- $\leq_{\text{m}}^2$ -complete partial function  $F$  maps some element of  $\gamma$ -**FP**  $\cap \text{dom } F$  to a  $\delta$ -**NP**- $\leq_{\text{m}}^1$ -complete element of  $Y$ .*

These lemmas will be used in Sections 4.2 and 4.3 to derive the non-uniform theorems from their uniform counterparts.

#### 4. APPLICATIONS

As noted in Section 3.1, our formulation can be viewed as a generalization of TTE achieved by removing the restrictions (a) and (b) on the oracle used as names. In the following three subsections, we will apply our theory to real numbers, real sets and real functions through representations  $\rho_{\mathbb{R}}$ ,  $\psi_{\odot}$ , and  $\delta_{\square}$ , which exploit the removal of (a), (b), and both, respectively.

For representations  $\gamma_0$  and  $\gamma_1$  of  $X_0$  and  $X_1$ , respectively, we can define the representation  $[\gamma_0, \gamma_1]$  of the Cartesian product  $X_0 \times X_1$  by  $[\gamma_0, \gamma_1](\langle \varphi_0, \varphi_1 \rangle) = (\gamma_0(\varphi_0), \gamma_1(\varphi_1))$ .

**4.1. Computation on real numbers.** Recall the representation  $\rho_{\mathbb{R}}$  of  $\mathbb{R}$  by infinite sequences (Section 2.1) where a  $\rho_{\mathbb{R}}$ -name of a real number  $x$  was a list  $u_0 \# u_1 \# \dots$  of rational numbers  $\llbracket u_i \rrbracket$  with  $|\llbracket u_i \rrbracket - x| < 2^{-i}$ . We adopt this in a straightforward way to define a representation  $\rho_{\mathbb{R}}$  that encodes real numbers by regular functions (we keep writing  $\rho_{\mathbb{R}}$  by abuse of notation): we say that  $\varphi \in \mathbf{Reg}$  is a  $\rho_{\mathbb{R}}$ -name of  $x \in \mathbb{R}$  if  $\varphi(0^i) \in \mathbf{D}$  and  $|\llbracket \varphi(0^i) \rrbracket - x| < 2^{-i}$  for each  $i \in \mathbb{N}$ . Thus we encode the same list in the values  $\varphi(0^i)$ .

We write  $\rho_{\mathbb{R}}|^{[0,1]}$  for the restriction of  $\rho_{\mathbb{R}}$  to (names of) real numbers in the interval  $[0, 1]$ . It is easy to see that the class  $(\rho_{\mathbb{R}}|^{[0,1]}, \rho_{\mathbb{R}})$ -**FP** coincides with the polynomial-time computability that was defined in Section 2.2.1 using the signed digit representation  $\rho_{\text{sd}}$ . Recall that in the definition of  $\rho_{\text{sd}}$ , we needed to forbid redundancy carefully. Now we do not have to worry too much about defining concise representations.

Moreover, we obtain a reasonable notion of polynomial-time computability of real functions  $f$  on  $\mathbb{R}$  (not just  $[0, 1]$ ) without additional work:  $(\rho_{\mathbb{R}}, \rho_{\mathbb{R}})$ -**FP** turns out to be a reasonable class that coincides with the one by Hoover [7], who required that the  $2^{-m}$ -approximation of the value  $f(t)$  should be delivered within time polynomial in  $m$  and  $\log|t|$  (this equivalence has been essentially observed by Lambov [18]). Another equivalent definition appears in Takeuti [21], inspired by Pour-El's approach to computable analysis.

*Example 4.1.* It is easy to see that binary addition and multiplication on  $\mathbb{R}$  are in  $([\rho_{\mathbb{R}}, \rho_{\mathbb{R}}], \rho_{\mathbb{R}})$ -**FP** by the algorithms suggested by Examples 2.2 and 2.3.

*Example 4.2.* The exponential function  $\exp: \mathbb{R} \rightarrow \mathbb{R}$  restricted to  $[0, 1]$  is in  $(\rho_{\mathbb{R}}|^{[0,1]}, \rho_{\mathbb{R}})$ -**FP**, because  $\exp t$  can be written as the sum of a series which is known to converge fast on  $[0, 1]$  (that is, given a desired precision, the machine can tell how many initial terms it needs to compute). However,  $\exp$  on the whole real line  $\mathbb{R}$  is not in  $(\rho_{\mathbb{R}}, \rho_{\mathbb{R}})$ -**FP**, because it grows too fast.

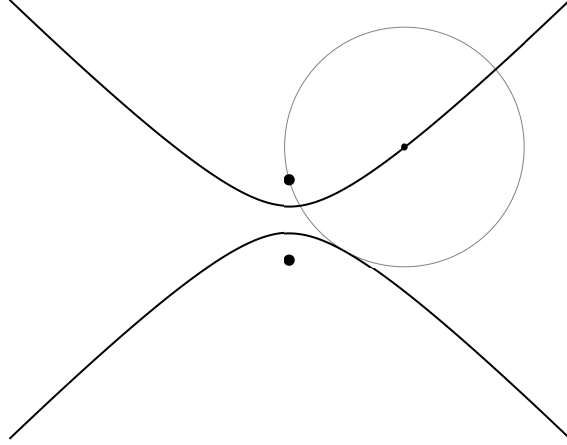


FIGURE 6. The trisector curves between two points.

*Example 4.3.* The sine function  $\sin: \mathbb{R} \rightarrow \mathbb{R}$  is in  $(\rho_{\mathbb{R}}, \rho_{\mathbb{R}})$ -**FP**. To see this, note that just like  $\exp$  in the previous example,  $\sin$  is polynomial-time computable if restricted to, say,  $[-4, 4]$ . It is also possible, given  $t \in \mathbb{R}$  and a desired precision, to find efficiently a number in  $[-4, 4]$  which is close enough to  $t$  modulo  $2\pi$ . We can compute  $\sin t$  by combining these algorithms.

*Example 4.4.* A function can belong to  $(\rho_{\mathbb{R}}, \rho_{\mathbb{R}})$ -**FP** without even an explicit description known. The *trisector curves* (Figure 6) between the points  $(0, 1)$  and  $(0, -1)$  in the plane are the pair of sets  $C_1, C_2 \subseteq \mathbb{R}^2$  such that  $C_1$  is the set of points equidistant from  $(0, 1)$  and  $C_2$ , and  $C_2$  is the set of points equidistant from  $(0, -1)$  and  $C_1$ . Asano, Matoušek and Tokuyama [1] proved that such a pair  $(C_1, C_2)$  exists and is unique (see [8] and [11] for simpler and more general proofs). They also showed that  $C_1$  (as well as  $C_2$ , which is its mirror image) is a graph of a function  $f: \mathbb{R} \rightarrow \mathbb{R}$  which is, in our terminology, in  $(\rho_{\mathbb{R}}, \rho_{\mathbb{R}})$ -**FP**. They conjecture that these curves are different from any curve that was previously known.

## 4.2. Computation on real sets.

**4.2.1. Representation of real sets.** Let  $\mathcal{A}$  be the set of closed subsets of  $[0, 1]^2$ . Define the representation  $\psi_{\odot}$  of  $\mathcal{A}$  as follows: let  $\varphi \in \mathbf{Pred}$  be a  $\psi_{\odot}$ -name of  $S \in \mathcal{A}$  if it satisfies the two itemized conditions in Section 2.2.3. Note that this representation is more succinct than the one that we would be able to define using infinite sequences [25, Example 6.9].

Since  $\text{dom } \psi_{\odot} \subseteq \mathbf{Pred}$ , it makes sense to talk about  $\psi_{\odot}$ -**NP** and  $(\psi_{\odot}, \psi_{\odot})$ -**NP** (Section 3.4.2). The following is immediate from the definition of polynomial-time computability in Section 2.2.3.

**Lemma 4.5.** *A set in  $\mathcal{A}$  is (nondeterministic) polynomial-time computable if and only if it is in  $\psi_{\odot}$ -P ( $\psi_{\odot}$ -NP).*

**4.2.2. Complexity of the convex hull operator.** The operator  $CH$  taking convex hulls (Section 2.2.3) is a function from  $\mathcal{A}$  to  $\mathcal{A}$ . We can state and prove the following uniform version of Theorems 2.6 and 2.7. As corollaries to this, we get Theorem 2.6 by Lemmas 3.11.2 and 4.5, and Theorem 2.7 by Lemmas 3.12.2 and 4.5.

**Theorem 4.6.**  *$CH$  is  $(\psi_{\odot}, \psi_{\odot})$ -NP- $\leq_m^2$ -complete.*

*Proof.* The main technical ideas are already in Ko and Yu's proof of the non-uniform versions (Theorems 2.6 and 2.7), so we will only sketch the proof.

That  $CH$  belongs to  $(\psi_{\odot}, \psi_{\odot})$ -**NP** is no surprise: A point  $p$  belongs to  $CH(S)$  if there are two points  $p'$  and  $p''$  in  $S$  such that  $p$  is on the line segment  $p'p''$ . All we have to do is to guess  $p'$  and  $p''$  nondeterministically, with appropriate consideration of precision.

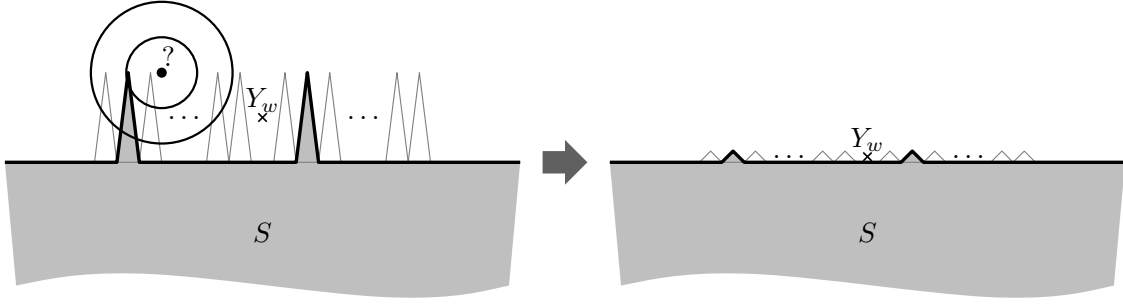


FIGURE 7. Widget for reducing **NP** to **CH**. We have  $Y_w \in CH(S)$  if and only if there is  $u$  such that the slot for  $(w, u)$  has a bump. In Ko and Yu's construction of  $S$ , the bumps can be high (left), and there can be a query that requires the knowledge of  $B(w, u)$  for many  $u$ . We make the bumps low (right) in order to make  $S$  polynomial-time computable in our sense.

For hardness, we need to modify the proof slightly, because, as we noted earlier, Ko and Yu's original results were about a weaker notion of computability: our computability of sets demands more in the sense that on query  $(u, v, 0^n)$ , where  $u, v \in \mathbf{D}$  and  $n \in \mathbb{N}$ , if  $(\llbracket u \rrbracket, \llbracket v \rrbracket)$  is within distance  $2^{-n}$  from the set, then we must say 1 (see the definition before Theorems 2.6 and 2.7), whereas for weak computability both 0 or 1 are allowed in this case.

We assume that the reader has Ko and Yu's proof [17, Corollary 4.6] at hand. The proof of their Lemma 4.4 begins by taking an arbitrary set  $A \in \mathbf{NP}$  and noting that there are  $B \in \mathbf{P}$  and a polynomial  $p$  such that  $w \in A$  if and only if  $(w, u) \in B$  for some string  $u$  of length exactly  $p(|w|)$ . Relativizing this, we take  $A \in \mathbf{NP}$ , and note that there are  $B \in \mathbf{P}$  and a second-order polynomial  $P$  such that  $A(\varphi)(w) = 1$  if and only if  $B(\varphi)(w, u) = 1$  for some string  $u$  of length  $P(|\varphi|)(|w|)$ .

We need to provide  $s$  and  $t$  of Definition 3.5 which reduce  $A$  to  $\psi_{\odot}^{-1} \circ CH \circ \psi_{\odot}$ . We define  $s$  by describing the set  $S = \psi_{\odot}(s(\varphi))$  for a given  $\varphi \in \mathbf{Reg}$ . The construction is similar to the original proof, replacing  $p(n)$  by  $P(|\varphi|)(n)$  and  $B$  by  $B(\varphi)$ .

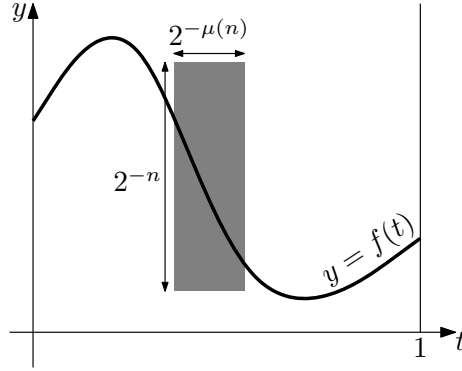
The original proof constructs, for each string  $w$ , the widget in Figure 7 left (or Figure 2 of [17]). In each of the left and right halves, there are exponentially many slots, one for each  $u$ , that have a bump if and only if  $(w, u)$  is in  $B$  (or  $B(\varphi)$  for us). The point of this construction is that, while the set  $S$  is easy to compute,  $CH(S)$  is hard in the sense that we can tell if  $w$  is in  $A$  (or  $A(\varphi)$ ) by checking whether the middle point  $Y_w$  belongs to  $CH(S)$ . But this  $S$  is not easy in our sense, because in order to answer the query shown in Figure 7, we need to know  $B(\varphi)(w, u)$  for exponentially many  $u$ . To avoid this, we make the bumps low, so they make an angle of at most  $45^\circ$  (Figure 7 right). This ensures that any one query to the  $\psi_{\odot}$ -name of  $S$  can be answered by checking  $B(\varphi)(w, u)$  for at most one  $(w, u)$ , making  $s$  computable in polynomial time.

The function  $t$  queries whether the point  $Y_w$  is in  $CH(S)$  with appropriate precision. Note that  $t$  needs access to  $\varphi$  in order to find the location of  $Y_w$  and the necessary precision.  $\square$

### 4.3. Computation on real functions.

**4.3.1. Representation of real functions.** We say that a non-decreasing function  $\mu: \mathbb{N} \rightarrow \mathbb{N}$  is a *modulus of continuity* of a function  $f \in C[0, 1]$  if for all  $n \in \mathbb{N}$  and  $t, t' \in [0, 1]$  such that  $|t - t'| \leq 2^{-\mu(n)}$ , we have  $|f(t) - f(t')| \leq 2^{-n}$  (Figure 8). Note that any  $f \in C[0, 1]$  is uniformly continuous and thus has a modulus of continuity.



FIGURE 8. Modulus of continuity  $\mu$  of a real function  $f \in C[0, 1]$ .

Define the representation  $\delta_\square$  of  $C[0, 1]$  as follows (see Lemmas 4.7 and 4.9 below for the reasons why we believe  $\delta_\square$  to be a natural representation): for  $\mu: \mathbb{N} \rightarrow \mathbb{N}$  and  $\varphi \in \mathbf{Reg}$ , we set  $\delta_\square(\langle \bar{\mu}, \varphi \rangle) = f$  if and only if  $\mu$  is a modulus of continuity of  $f$  and for every  $n \in \mathbb{N}$  and  $u \in \mathbf{D}$ , we have  $v := \varphi(0^n, u) \in \mathbf{D}$  and  $|f(\llbracket u \rrbracket) - \llbracket v \rrbracket| < 2^{-n}$  (the string  $v$  may have to have leading 0s padded in order to make  $\varphi$  regular—but this need for padding is inconsequential, see the penultimate paragraph of Section 3.2; in what follows, we omit this padding in the description of algorithms). To see that  $\delta_\square(\varphi)$  is well-defined, suppose that the above condition holds for two real functions  $f$  and  $f'$ . Let  $t \in [0, 1]$  be arbitrary. Then for each  $n \in \mathbb{N}$ , there is  $u \in \mathbf{D}$  with  $|t - \llbracket u \rrbracket| \leq 2^{-\mu(n)}$  and thus

$$\begin{aligned}
 (11) \quad |f(t) - f'(t)| &\leq |f(t) - f(\llbracket u \rrbracket)| + |f(\llbracket u \rrbracket) - \llbracket \varphi(0^n, u) \rrbracket| \\
 &\quad + |f'(\llbracket u \rrbracket) - \llbracket \varphi(0^n, u) \rrbracket| + |f'(t) - f'(\llbracket u \rrbracket)| \\
 &\leq 2^{-n} + 2^{-n} + 2^{-n} + 2^{-n} = 2^{-n+2}.
 \end{aligned}$$

Since  $n \in \mathbb{N}$  was arbitrary,  $f(t) = f'(t)$ . Since  $t \in [0, 1]$  was arbitrary,  $f = f'$ .

Recall that the only reason that a real number can require long  $\rho_{\mathbb{R}}$ -names was having a large absolute value. In contrast, functions in  $C[0, 1]$  may have long  $\delta_\square$ -names for two possible reasons: having big values, and having a big modulus of continuity.

The following lemma says that the complexity of  $\delta_\square$ -names of  $f \in C[0, 1]$  matches the complexity of  $f$  that was discussed in Section 4.1 using the representation  $\rho_{\mathbb{R}}$ :

**Lemma 4.7** ([14, Corollary 2.21]). *A function in  $C[0, 1]$  is polynomial-time (resp. polynomial-space) computable if and only if it is in  $\delta_\square$ -FP (resp.  $\delta_\square$ -FPSPACE).*

**Lemma 4.8.** *A function in  $C[0, 1]$  is PSPACE-complete in the sense of [10, Section 2.2] if it is  $\delta_\square$ -FPSPACE- $\leq_{\text{mF}}^1$ -complete and has a polynomial modulus of continuity.*

*Proof.* Suppose that  $f \in C[0, 1]$  is  $\delta_\square$ -FPSPACE- $\leq_{\text{mF}}^1$ -complete and has a polynomial modulus of continuity  $\mu$ . Then for any  $A \in \mathbf{PSPACE}$  there are polynomial-time functions  $t$  and  $r$  that satisfy the first picture of Figure 4 for any  $B \in \delta_\square^{-1}[f]$ —and thus for any  $B$  of the form  $\langle \bar{\mu}, \varphi \rangle$  (that is, those with this particular polynomial  $\mu$  in the first component). A query to  $B$  can ask either a value of  $\mu$  or a value of  $\varphi$ , but  $\mu$  is just a polynomial, so we may assume that  $t$  only asks a query of form “ $\varphi(0^n, v)$ ?”. Thus, given  $u$ , an answer in  $A[u]$  can be computed by  $r$  from  $u$  and a  $2^{-n}$ -approximation of  $f(\llbracket v \rrbracket)$ . This implies that  $f$  is PSPACE-complete.  $\square$

The representation  $\delta_\square$  of  $C[0, 1]$  may look somewhat arbitrary at first sight. Here we present a property of  $\delta_\square$  that seems to make it a “natural” representation. Define the function *Apply*:  $C[0, 1] \times [0, 1] \rightarrow \mathbb{R}$  by  $\text{Apply}(f, x) = f(x)$ . The following lemma says that (the  $\equiv$ -equivalence class of) the representation  $\delta_\square$  is the least informative representation of

$C[0, 1]$  that makes *Apply* efficiently computable (see Section 3.4.1 for the definitions of  $\leq$  and  $\equiv$ ). The proof will appear in a forthcoming paper.

**Lemma 4.9.** *Let  $\delta$  be any representation of  $C[0, 1]$ . Then  $\text{Apply} \in ([\delta, \rho_{\mathbb{R}}|^{[0,1]}, \rho_{\mathbb{R}})\text{-FP}$  if and only if  $\delta \leq \delta_{\square}$ .*

The above definitions and lemmas extend to some well-behaved compact domains other than  $[0, 1]$  (we keep writing  $\delta_{\square}$  by abuse of notation). To discuss the complexity of the operator *LipIVP* (Section 2.2.3), we define a representation  $\delta_{\square L}$  of the space  $\text{CL}[[0, 1] \times [-1, 1]]$  of Lipschitz continuous functions by setting  $\delta_{\square L}(\langle \varphi, 0^L \rangle) = g$  if and only if  $\varphi$  is a  $\delta_{\square}$ -name of  $g$  and  $L \in \mathbb{N}$  satisfies (4) (regard the string  $0^L$  as the constant function whose value is  $0^L$ ).

**4.3.2. Complexity of the operator that solves differential equations.** Now we can formulate the uniform version of Theorems 2.8 and 2.9 as follows (a proof will be given shortly).

**Theorem 4.10.**  *$\text{LipIVP}$  is  $(\delta_{\square L}, \delta_{\square})\text{-FPSPACE-}\leq_{\text{mF}}^2\text{-complete}$ .*

As corollaries, we have Theorem 2.8 by Lemmas 3.11.1 and 4.7, and Theorem 2.9 by Lemmas 3.12.1 and 4.8.

The following weaker version of Theorem 4.10, stated with the stronger reduction  $\leq_{\text{W}}^2$  from Definition 3.5.3, is slightly easier to prove (see the end of the section):

**Corollary 4.11.**  *$\text{LipIVP}$  is  $(\delta_{\square L}, \delta_{\square})\text{-FPSPACE-}\leq_{\text{W}}^2\text{-complete}$ .*

As we noted in Lemma 3.10, this is a more robust result in the sense that it is invariant under replacing representations to  $\equiv$ -equivalent ones. A drawback is that Corollary 4.11 does not directly yield Theorem 2.9, because Lemma 4.8 requires  $\text{FPSPACE-}\leq_{\text{mF}}^1\text{-completeness}$ , whereas replacing  $\leq_{\text{mF}}^2$  by  $\leq_{\text{W}}^2$  in the assumption of 3.12.1 also changes  $\leq_{\text{mF}}^1$  to  $\leq_{\text{T}}^1$  in the conclusion. We can still obtain Corollary 2.10.

The rest of the section is devoted to the proof of Theorem 4.10. The positive part ( $\text{LipIVP} \in (\delta_{\square L}, \delta_{\square})\text{-FPSPACE}$ ) will be verified by checking that the proof of Theorem 2.8 can be made uniform. For the hardness, we need to modify slightly the construction in the original proof of Theorem 2.9 (this modification is not needed if we only want Corollary 4.11).

*Proof of Theorem 4.10, computability.* Given a  $\delta_{\square L}$ -name  $\langle \bar{\mu}, \varphi, 0^L \rangle$  of  $g$ , we need to find a  $\delta_{\square}$ -name  $\langle \bar{\nu}, \psi \rangle$  of  $h = \text{LipIVP}(g)$ . Recall that  $\mu$  is a modulus of continuity of  $g$ , and  $|\llbracket \varphi(0^q, u, v) \rrbracket - f(\llbracket u \rrbracket, \llbracket v \rrbracket)| < 2^{-q}$  for each  $u, v \in \mathbf{D}$  (such that  $(\llbracket u \rrbracket, \llbracket v \rrbracket) \in [0, 1] \times [-1, 1]$ ).

It is easy to find a modulus of continuity  $\nu$  of  $h$ : let  $\nu(n) = n + M$ , where  $M \in \mathbb{N}$  is any number such that the values of  $g$  always stay in  $[-2^M, 2^M]$ . For example,  $M = \lceil \log_2(|\llbracket \varphi(\varepsilon, +0/1, +0/1) \rrbracket| + 1 + 2^{\mu(0)}) \rceil$ .

To obtain  $\psi$ , we apply the forward Euler method with step size  $2^{-p}$  to the approximation of  $g$  with precision  $2^{-q}$  (we will specify  $p$  and  $q$  shortly). That is, we define an approximation  $\tilde{h}_{p,q} \in C[0, 1]$  of  $h$  by letting  $\tilde{h}_{p,q}(0) = 0$  and then defining  $\tilde{h}_{p,q}$  on  $[2^{-p}T, 2^{-p}(T+1)]$ , for each  $T = 0, \dots, 2^p - 1$  inductively, to be linear with slope approximately  $g(2^{-p}T, h(2^{-p}T))$ : formally,

$$(12) \quad \tilde{h}_{p,q}(2^{-p}T + \Delta t) = \tilde{h}_{p,q}(2^{-p}T) + \Delta t \llbracket \varphi(0^q, u, v) \rrbracket, \quad 0 \leq \Delta t \leq 2^{-p},$$

for some  $u, v \in \mathbf{D}$  with  $\llbracket u \rrbracket = 2^{-p}T$  and  $\llbracket v \rrbracket = \tilde{h}_{p,q}(2^{-p}T)$ . Obviously, we can compute such a function  $\tilde{h}_{p,q}$  in space polynomial in  $p$  and  $q$  in the sense that there is  $\text{Euler} \in \text{FPSPACE}$  such that  $\llbracket \text{Euler}(\varphi)(0^p, 0^q, u) \rrbracket = \tilde{h}_{p,q}(\llbracket u \rrbracket)$  for every  $u \in \mathbf{D}$ .

Let  $\psi(0^n, u) = \text{Euler}(\varphi)(0^p, 0^q, u)$ , where  $p = \max\{\mu(n+8L), n+8L+M\}$  and  $q = n+8L$ . We claim that  $\langle \bar{\nu}, \psi \rangle$  is a  $\delta_{\square}$ -name of  $h$  (this proves the desired  $\text{FPSPACE}$ -computability, since

$p$  and  $q$  are bounded polynomially in  $|\varphi|$ ,  $\mu$  and  $n, L$ ). This means that  $|\tilde{h}_{p,q}(t) - h(t)| \leq 2^{-n}$  for any  $t \in [0, 1]$ . More strongly, we prove, by induction on  $T = 0, \dots, 2^p - 1$ , that

$$(13) \quad |\tilde{h}_{p,q}(t) - h(t)| \leq 2^{-n} e^{4L(t-1)}$$

for all  $t \in [2^{-p}T, 2^{-p}(T+1)]$ . We may assume (13) for  $t = 2^{-p}T$  as the induction hypothesis. The approximate value  $\tilde{h}_{p,q}(2^{-p}T + \Delta t)$  is defined by (12), whereas the true solution  $h$  satisfies

$$(14) \quad h(2^{-p}T + \Delta t) = h(2^{-p}T) + \int_{2^{-p}T}^{2^{-p}T + \Delta t} g(\tau, h(\tau)) d\tau.$$

The error added by this approximation is

$$(15) \quad \left| \Delta t \llbracket \varphi(0^q, u, v) \rrbracket - \int_{2^{-p}T}^{2^{-p}T + \Delta t} g(\tau, h(\tau)) d\tau \right| \leq 4L2^{-n} e^{4L(2^{-p}T-1)} \Delta t,$$

because

$$(16) \quad \begin{aligned} & \left| \llbracket \varphi(0^q, u, v) \rrbracket - g(\tau, h(\tau)) \right| \\ & \leq \left| \llbracket \varphi(0^q, u, v) \rrbracket - g(\llbracket u \rrbracket, \llbracket v \rrbracket) \right| + \left| g(\llbracket u \rrbracket, \llbracket v \rrbracket) - g(\tau, \llbracket v \rrbracket) \right| + \left| g(\tau, \llbracket v \rrbracket) - g(\tau, h(\tau)) \right| \\ & \leq 2^{-q} + 2^{-n-8L} + L|\llbracket v \rrbracket - h(\tau)| \\ & \leq 2^{-n-8L} + 2^{-n-8L} + L(|\llbracket v \rrbracket - h(2^{-p}T)| + |h(2^{-p}T) - h(\tau)|) \\ & \leq 2^{-n-8L} + 2^{-n-8L} + L(2^{-n} e^{4L(2^{-p}T-1)} + 2^{-p}2^M) \\ & \leq L(2^{-n-8L} + 2^{-n-8L} + 2^{-n} e^{4L(2^{-p}T-1)} + 2^{-n-8L}) \leq 4L2^{-n} e^{4L(2^{-p}T-1)}, \end{aligned}$$

where the second, third and fifth inequalities come from  $p \geq \mu(n + 8L)$ ,  $q \geq n + 8L$ ,  $p \geq M + n + 8L$ , respectively. Using (15) and the induction hypothesis, we compare (12) and (14) to obtain

$$(17) \quad \begin{aligned} |\tilde{h}_{p,q}(2^{-p}T + \Delta t) - h(2^{-p}T + \Delta t)| & \leq 2^{-n} e^{4L(2^{-p}T-1)} + 4L2^{-n} e^{4L(2^{-p}T-1)} \Delta t \\ & = 2^{-n} e^{4L(2^{-p}T-1)} (1 + 4L\Delta t) \leq 2^{-n} e^{4L(2^{-p}T-1)} e^{4L\Delta t} = 2^{-n} e^{4L(2^{-p}T-1+\Delta t)}, \end{aligned}$$

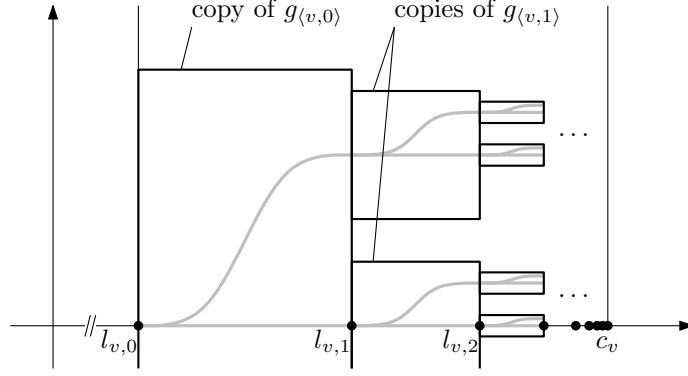
as desired.  $\square$

For the hardness, the core part of the proof can be done by relativizing the argument for the non-uniform version [10]. Since the proof was by reduction from the problem QBF, we use the relativized version QBF<sup>2</sup> from Lemma 3.8. Starting from QBF<sup>2</sup>, we follow the construction in [10, Lemma 4.1], which uniformizes and yields the following. Let  $\iota_{\Sigma^*}$  be the representation of  $\Sigma^*$  which encodes a finite string  $u$  by the constant function with value  $u$ . Let  $\Lambda$  be the set of non-decreasing functions from  $\mathbb{N}$  to  $\mathbb{N}$ , and let  $\iota_{\Lambda}$  be its representation defined by  $\iota_{\Lambda}(\varphi) = \overline{\varphi}$ .

**Lemma 4.12.** *Let  $L \in \mathbf{PSPACE}$ . Then there are a second-order polynomial  $P$  and a function  $G \in ([\text{id}, \iota_{\Lambda}, \iota_{\Sigma^*}, \rho_{\mathbb{R}}]^{[0,1]}, \rho_{\mathbb{R}}]^{[-1,1]}, \rho_{\mathbb{R}}) \text{-FP}$  such that for each  $\varphi \in \text{dom } L$ ,  $\lambda \in \Lambda$  and  $u \in \Sigma^*$ , the function  $g_u^{\varphi, \lambda}: [0, 1] \times [-1, 1] \rightarrow \mathbb{R}$  defined by  $g_u^{\varphi, \lambda}(t, y) = G(\varphi, \lambda, u, t, y)$  satisfies*

- (1)  $g_u^{\varphi, \lambda}(0, y) = g_u^{\varphi, \lambda}(1, y) = 0$  for all  $y \in [-1, 1]$ ;
- (2)  $|g_u^{\varphi, \lambda}(t, y_0) - g_u^{\varphi, \lambda}(t, y_1)| \leq 2^{-\lambda(|u|)} |y_0 - y_1|$  for any  $t \in [0, 1]$  and  $y_0, y_1 \in [-1, 1]$ ;
- (3)  $g_u^{\varphi, \lambda}(t, y) \in \text{dom } \text{LipIVP}$ , and  $h_u^{\varphi, \lambda} := \text{LipIVP}(g_u^{\varphi, \lambda})$  satisfies  $h_u^{\varphi, \lambda}(1) = 2^{-P(|\varphi|, \lambda)(|u|)} \cdot L(\varphi)(u)$ .

*Proof of Theorem 4.10, hardness.* Let  $F \in \mathbf{FPSPACE}$ . We need to show that  $F \leq_{\text{mF}}^2 \delta_{\square}^{-1} \circ \text{LipIVP} \circ \delta_{\square\text{L}}$ . We may assume that  $F$  is a total function and that there is a second-order polynomial  $Q$  such that  $F(\varphi)(v)$  has length exactly  $Q(|\varphi|)(|v|)$  for all  $\varphi \in \mathbf{Reg}$  and

FIGURE 9. Widget for reducing **FSPACE** to *LipIVP*.

$v \in \Sigma^*$ . There is  $L \in \mathbf{PSPACE}$  such that  $L(\varphi)(v, 0^i)$  equals the  $i$ th symbol of  $F(\varphi)(v)$  for any  $\varphi \in \mathbf{Reg}$ ,  $v \in \Sigma^*$  and  $i \in \{0, 1, \dots, Q(|\varphi|)(|v|) - 1\}$ . Apply Lemma 4.12 to this  $L$  to obtain the  $P$  and  $G$ , and let  $g_u^{\varphi, \lambda}$  and  $h_u^{\varphi, \lambda}$  be as in the Lemma.

We define  $s$  (of Definition 3.5) by describing the real function  $g = \delta_{\square L}(s(\varphi)) \in \text{CL}[[0, 1] \times [-1, 1]]$  for a given  $\varphi$ . It has Lipschitz constant 1. It will be straightforward to check that a  $\delta_{\square}$ -name (and hence a  $\delta_{\square L}$ -name) of  $g$  can be **FP**-computed from  $\varphi$ . We write  $g_u$  and  $h_u$  for the  $g_u^{\varphi, \lambda}$  and  $h_u^{\varphi, \lambda}$  corresponding to this given  $\varphi$  and  $\lambda(k) = 3k + 2$ .

For each binary string  $v$ , let

$$(18) \quad c_v = 1 - \frac{1}{2^{|v|}} + \frac{2\bar{v} + 1}{2^{2|v|+2}}, \quad l_v^{\mp} = c_v \mp \frac{1}{2^{2|v|+2}},$$

where  $\bar{v} \in \{0, \dots, 2^{|v|} - 1\}$  means  $v$  interpreted as an integer in binary notation. This divides  $[0, 1)$  into intervals  $[l_v^-, l_v^+]$  indexed by  $v \in \{0, 1\}^*$ . We further divide the left half  $[l_v^-, c_v]$  into  $Q(|\varphi|)(|v|) + 1$  subintervals  $[l_{v,0}, l_{v,1}]$ ,  $[l_{v,1}, l_{v,2}]$ ,  $\dots$ ,  $[l_{v,Q(|\varphi|)(|v|)-1}, l_{v,Q(|\varphi|)(|v|)}]$ ,  $[l_{v,Q(|\varphi|)(|v|)}, c_v]$ , where

$$(19) \quad l_{v,i} = c_v - \frac{1}{2^{2|v|+2}2^i}, \quad i = 0, 1, \dots, Q(|\varphi|)(|v|).$$

On each strip  $[l_{v,i}, l_{v,i+1}] \times [-1, 1]$ , we define  $g$  by putting the copies of  $g_{(v,0^i)}$  as in Figure 9. Precisely,

$$(20) \quad g\left(l_{v,i} + \frac{t}{2^{2|v|+2}2^{i+1}}, \frac{2m + (-1)^m y}{2^{\gamma(|v|,i)}}\right) = \frac{2^{2|v|+2}2^{i+1}}{2^{\gamma(|v|,i)}} g_{(v,0^i)}(t, y)$$

for each  $t \in [0, 1]$  and  $m \in \mathbb{N}$ ,  $y \in [-1, 1]$ , where the polynomial  $\gamma$  is defined by  $\gamma(|v|, 0) = 2|v| + 3$  and  $\gamma(|v|, i+1) = \gamma(|v|, i) + P(|\varphi|, \lambda)(|(v, 0^i)|) + 2$ . On the last strip  $[l_{v,Q(|\varphi|)(|v|)}, c_v] \times [-1, 1]$ , we define  $g$  to be constantly 0. On the right half  $[c_v, l_v^+]$ , we define  $g$  symmetrically:  $g(l^+ - t, y) = -g(l^- + t, y)$  for  $0 \leq t \leq 1/2^{2|v|+2}$ . Because of this symmetry, the function  $h := \text{LipIVP}(g)$  takes value 0 at each  $l_v^{\mp}$ , and it can be verified, using (3) of Lemma 4.12, that for  $i = 0, \dots, Q(|\varphi|)(|v|)$  inductively,

$$h(l_{v,i}) = \sum_{j=0}^{i-1} \frac{h_{(v,0^j)}(1)}{2^{\gamma(|v|,j)}} = \sum_{j=0}^{i-1} \frac{L(\varphi)(v, 0^j)}{2^{\gamma(|v|,j)} 2^{P(|\varphi|, \lambda)(|(v, 0^j)|)}} = \sum_{j=0}^{i-1} \frac{4L(\varphi)(v, 0^j)}{2^{\gamma(|v|,j+1)}}.$$

In particular, the number  $h(c_v) = h(l_{v,Q(|\varphi|)(|v|)}) = \sum_{j=0}^{Q(|\varphi|)(|v|)-1} 4L(\varphi)(v, 0^j) / 2^{\gamma(|v|,j+1)}$  contains the values  $L(\varphi)(v, 0^j)$  for all  $j < Q(|\varphi|)(|v|)$ , from which we can recover  $F(\varphi)(v)$ . The reducing functions  $r$  and  $t$  (of Definition 3.5) perform this lookup. That is,  $t(\varphi)(v) =$

$(0^{\gamma(|v|, Q(|\varphi|)(|v|))}, w)$  with  $\llbracket w \rrbracket = c_v$ , and  $r(\varphi)$  is the function that, given the encoding of (an approximation of)  $h(c_v)$ , extracts the value  $F(\varphi)(v)$ .  $\square$

In [10, Theorem 3.2], the non-uniform version of Lemma 4.12 was used to construct a function that proved Theorem 2.9. We needed a different construction, because for our Theorem 4.10 (with the reduction  $\leq_{\text{mF}}^2$ ), we needed to get the values  $L(\varphi)(v, 0^j)$  for all  $j < Q(|\varphi|)(|v|)$  in one query. For Corollary 4.11 (with the reduction  $\leq_{\text{W}}^2$ ), we are allowed to make many queries, so the straightforward uniformization (without stacking the copies of  $g_{(v, 0^i)}$  vertically) would have worked.

## 5. SUMMARY AND FUTURE WORK

- To discuss computational complexity in the framework of TTE, we replace  $\Sigma^{\mathbb{N}}$ , the infinite strings, by **Reg**, a class of functions from strings to strings. This is a generalization in two ways: these functions (a) can have values of arbitrary length, and (b) take string arguments, rather than just unary strings.
- For time and space bounds we use second-order polynomials in the input size, which are defined in the way suggested by type-two complexity theory. We defined classes **P**, **NP** and **FP**, **FPSPACE**. With a suitable notion of polynomial-time reductions, we can also define **NP**- and **FPSPACE**-completeness. Formulating other classes is left for future work.
- To apply this to problems involving real numbers, we introduced representations  $\rho_{\mathbb{R}}$ ,  $\psi_{\odot}$  and  $\delta_{\square}$  of real numbers, sets and functions. Both aspects (a) and (b) of our generalization were useful. With respect to these representations, we showed that taking the convex hull of a set is **NP**-complete, and that solving a Lipschitz continuous ordinary differential equation is **FPSPACE**-complete. These are uniform versions of what have been known non-uniformly, and tell us more about the hardness of numerical problems in practice. An interesting direction for further investigation is to ask which other known non-uniform results about operators do (or do not) uniformize. One can also look at known computability results and ask whether analogous statements hold true for time- or space-bounded classes.

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## REFERENCES

- [1] T. Asano, J. Matoušek, and T. Tokuyama. The distance trisector curve. *Adv. Math.*, 212(1):338–360, 2007.
- [2] P. Beame, S. Cook, J. Edmonds, R. Impagliazzo, and T. Pitassi. The relative complexity of NP search problems. *J. Comput. Syst. Sci.*, 57(1):3–19, 1998.
- [3] V. Brattka, P. Hertling, and K. Weihrauch. A tutorial on computable analysis. In S. B. Cooper, B. Löwe, and A. Sorbi, editors, *New Computational Paradigms: Changing Conceptions of What is Computable*, pages 425–491. Springer, 2008.
- [4] V. Brattka and G. Gherardi. Weihrauch Degrees, Omniscience Principles and Weak Computability. *J. Symbolic Logic*, 76(1):143–176, 2011.

- [5] M. Braverman. On the complexity of real functions. In *Proc. 46th Annual IEEE Symposium on Foundations of Computer Science*, pages 155–164, 2005.
- [6] O. Goldreich. *Computational Complexity: A Conceptual Perspective*. Cambridge University Press, 2008.
- [7] H. J. Hoover. Feasible real functions and arithmetic circuits. *SIAM J. Comput.*, 19(1):182–204, 1990.
- [8] K. Imai, A. Kawamura, J. Matoušek, D. Reem, and T. Tokuyama. Distance  $k$ -sectors exist. *Comput. Geom.*, 43(9):713–720, 2010.
- [9] B. M. Kapron and S. A. Cook. A new characterization of type-2 feasibility. *SIAM J. Comput.*, 25(1):117–132, 1996.
- [10] A. Kawamura. Lipschitz continuous ordinary differential equations are polynomial-space complete. *Comput. Complexity*, 19(2):305–332, 2010.
- [11] A. Kawamura, J. Matoušek, and T. Tokuyama. Zone diagrams in Euclidean spaces and in other normed spaces. *Math. Annal.*, in press.
- [12] K. Ko and H. Friedman. Computational complexity of real functions. *Theoret. Comput. Sci.*, 20(3):323–352, 1982.
- [13] K. Ko. On the computational complexity of ordinary differential equations. *Inform. Contr.*, 58:157–194, 1983.
- [14] K. Ko. *Complexity Theory of Real Functions*. Birkhäuser Boston, 1991.
- [15] K. Ko. On the computational complexity of integral equations. *Ann. Pure Appl. Log.*, 58(3):201–228, 1992.
- [16] K. Ko. Polynomial-time computability in analysis. In I. L. Ershov et al., editors, *Handbook of Recursive Mathematics: Volume 2: Recursive Algebra, Analysis and Combinatorics*, vol. 139 of *Studies in Logic and the Foundations of Mathematics*, pages 1271–1317. North-Holland, 1998.
- [17] K. Ko and F. Yu. On the complexity of convex hulls of subsets of the two-dimensional plane. In *Proc. 4th International Conference on Computability and Complexity in Analysis*, vol. 202 of *Electronic Notes in Theoretical Computer Science*, pages 121–135, 2008.
- [18] B. Lambov. The basic feasible functionals in computable analysis. *J. Complexity*, 22(6):909–917, 2006.
- [19] K. Mehlhorn. Polynomial and abstract subrecursive classes. *J. Comput. Syst. Sci.*, 12(2):147–178, 1976.
- [20] H. Ota, A. Kawamura, M. Ziegler, and C. Rösnick. Complexity of smooth ordinary differential equations. Presented at the 10th EATCS/LA Workshop on Theoretical Computer Science. In Japanese.
- [21] I. Takeuti. Effective fixed point theorem over a non-computably separable metric space. In J. Blanck, V. Brattka, and P. Hertling, editors, *Computability and Complexity in Analysis*, vol. 2064 of *Lecture Notes in Computer Science*, pages 310–322, 2001.
- [22] I. Wegener. *Komplexitätstheorie: Grenzen der Effizienz von Algorithmen*. Springer, 2003. In German.
- [23] K. Weihrauch. The degrees of discontinuity of some translators between representations of the real numbers. Technical Report TR-92-050, International Computer Science Institute, Berkeley, 1992.
- [24] K. Weihrauch. *Computable Analysis: An Introduction*. Texts in Theoretical Computer Science. Springer, 2000.
- [25] K. Weihrauch. Computational complexity on computable metric spaces. *Math. Log. Q.*, 49(1):3–21, 2003.
- [26] X. Zhao and N. Müller. Complexity of operators on compact sets. In *Proc. 4th International Conference on Computability and Complexity in Analysis*, vol. 202 of *Electronic Notes in Theoretical Computer Science*, pages 101–119, 2008.

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